Investigation of Time-Dependent Microscale Close Contact Melting

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Close-contact melting process occurs due to direct heating of phase change material (PCM) by a sliding heater plate. Due to the shearing motion, a squeeze film flow is developed between them, which generates the pressure needed to support the PCM. Such close-contact melting phenomena have numerous applications in engineering and natural settings. When a micro flow is developed in the squeeze film, for example in microelectromechanical systems (MEMS), slip effects in velocity and/or temperature can become significant. This study investigates the influence of slip velocity and/or temperature on the contact melting of an electrically conducting PCM in the presence of a magnetic field. An analytical solution for the thin film flow and energy transport coupled with unsteady phase change heat transfer under Navier slip conditions and their interactions with the electromagnetic fields specified via the Maxwell's equations is developed. Numerical solutions of the resulting model in non-dimensional form revealed the effect of various characteristic parameters on the transient variation of the melting rate and the liquid film thickness. In particular, it is found that increasing the slip velocity in the absence of temperature slip increases the melting rate, with an earlier onset of the steady state. More specifically, the melting rate increases by 24% and the film thickness decreases by 28% when the dimensionless slip length for velocity λ equals to 1 × 10−3 compared to the corresponding no slip case. On the other hand, when temperature slip is included, an increase in slip velocity leads to slower melting rate as well as taking longer to attain the steady state. The melting rate decreases 36% and the film thickness increases by 39%, when the dimensionless slip length for temperature λθ equals to 1 × 10−3 compared to corresponding no slip case.

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1. Introduction

Pressing a solid block of a phase change material (PCM) at its phase transition temperature against a heated plate causes melting of the PCM. The result is a thin liquid film formed between the solid and the heated plate which is continuously squeezed by the solid descending downward. Close contact melting is a phenomenon characterized by thin liquid film hydrodynamics and energy transport with phase change heat transfer and involving the dynamics of the solid and heated plate. Close contact melting has several applications in nature and technology, manufacturing process [1] (e.g., welding and hot wire cutting), nuclear technology [2], latent heat thermal storage system [3,4], Leidenfrost effect[5], interior ballistics[6] and melt lubrication and burial of heat generating bodies as well as ice skating [7]. Moallemi et al [8] carried out one of the earliest studies in this area including experimental and analytical investigation involving solid n-octadecane melting by a horizontal planar heat source at constant surface temperature. Bejan [7] presented a quasi-steady mathematical model of contact melting due to direct heating by conduction as well as that due to frictional heating. Also, he presented detailed reviews on contact melting heat transfer under different configurations [9,10]. All these previous studies concerned mainly on determining the quasi-steady characteristics (e.g., melt rate and the liquid film thickness) of the contact-melting process and few investigations on its transient characteristics are available in the literature. For instance, Hong and Saito [11] performed a simplified numerical study on the transient features of the contact melting problem under constant wall temperature, which was extended to include the constant wall heat flux condition by Saito et al [12]. Later, an analytical formulation under both the wall heating conditions that yielded basic insights into the transient dynamics of the contact-melting heat transfer process was reported in [13,14], and a Lagrangian approach of spatially varying heat flux in close
contact melting was described by [15]. Soni and Premnath [16] described the magnetic field effects on the natural convection from a vertical melting substrate, i.e., without close-contact melting pro-

cess, and lately [17] investigated the effect of the magnetic field on direct contact melting transport process during rotation using a finite difference scheme. The evolution of flow and heat transfer of PCM due to natural convection melting in a square cavity with a local heater was reported in [18,19]. An experimental study of close contact melting heat transfer on a heated horizontal plate in the presence of nano-enhanced PCM was presented by [20]. More recently, Ghalambaz et al. [21] studied free convection heat transfer of a suspension of Nano–Encapsulated Phase Change Materials (NEPCMs) in an inclined porous cavity and found that the presence of NEPCMs leads to improvement in heat transfer. Mehryan et al. [22] presented phase change Heat Transfer in an inclined compound cavity partially filled with a porous medium and concluded that the melting rate is higher when the inclination angle is 45 degree and also when the thickness of the porous layer increases. In another study [23], they also reported the investigation of the thermal performance of new types of hybrid nanoparticles and encapsulated phase change particles. Sadeghi et al. [24] investigated the use of multi-layer PCMs in the tubular heat exchanger with periodic heat transfer boundary condition. Recently, Turkylmazoglu [25] performed an analysis of the direct contact melting due to a permeable rotating disk, while a mathematical formulation was developed and a study of the effect of temperature dependent properties on contact melting was performed in a latest work [26].

None of the previous investigations considered the effect of slip velocity or slip temperature on the unsteady sliding contact melting heat transfer, especially when subjected to a magnetic field. Considerations of slip conditions on heater plate surfaces are of fundamental interest in microelectromechanical system (MEMS) involving micro flows accompanied by phase change processes. Moreover, the interactions between the thin film flow and electromagnetic fields induce the Lorentz body force which can be used to manipulate the dynamics of the contact–melting process in such configurations. In the present work, the effect of slip in velocity and/or temperature on the sliding contact-melting phenomenon is studied. In this regard, an analytical solution is presented which describes the transient characteristics of the contact melting process (e.g., melt velocity and liquid film thickness) over slip walls when subjected to a magnetic field. This is achieved by developing a mathematical model representing the thin film hydrodynamics under the lubrication approximation with the Navier slip condition and subjected to the Lorentz force, and along with energy equation, Maxwell’s equations for the induced magnetic fields and the Stefan condition representing the energy balance at the solid–liquid interface.

2. Physical Model

A schematic explanation of physical system is shown in Fig. 1. A rectangular parallelepiped solid block of size \( L \times W \times H \) with its melting temperature \( T_{m} \) is contacting a hot plate at temperature \( T_{w} (T_{w} > T_{m}) \) by a vertical force (its own weight). An external magnetic field \( B_{0} \) is applied normal to the direction of the motion of the heated plate (see Fig. 1). The solid block (PCM), which is electrically conducting, starts to melt due to a horizontal tangential force \( F_{x} \) from time \( t = 0 \) creating a thin liquid film flow on the heated plate which moves at constant relative velocity \( U \) under slip velocity conditions in the \( x \) direction. As contact is kept by the externally applied force on the solid block, considerations of gravity will be similar to that presented in [9,13]. The block of PCM moves vertically downward with the descending velocity \( V \) due to melting and producing an electrically conducting thin liquid film of thickness, \( \delta \), covering the swept portion of the moving surface. Both solid descending velocity, \( V \), and liquid film thickness, \( \delta \), change with time, and ultimately reach a quasi-steady state, where
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Aljaghtham, rize by This much to consideration of the solid mass is assumed to be much smaller than the gravity, \( \frac{\partial T}{\partial z} \approx g \). Hence, the term \( \frac{\partial T}{\partial z} \) is neglected [11]. In addition, the Stefan condition representing the energy balance between the solid block (i.e., PCM) and the liquid film is as follows:

\[
-k \left( \frac{\partial T}{\partial z} \right)_{z=\delta} = \rho_s h_{df} \left( V + \frac{\partial \delta}{\partial t} \right), \tag{2}
\]

where \( k \), \( \rho_s \) and \( h_{df} \) represent the thermal conductivity, density of the solid block and latent heat of fusion, respectively. The rate of descending movement of the solid, \( V \), and the rate of increase in liquid film thickness, \( \delta(t) \), the tangential force, \( F_t \), can be determined from the total viscous shear force in the liquid film which is expressed as [10,11]:

\[
F_x = -\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{0}^{\frac{1}{2}} \mu \left( \frac{\partial u}{\partial z} \right)_{z=0} \, dx \, dy. \tag{3}
\]

2.1. Theoretical Analysis

2.1.1. Thin Film Flow Equations

We now consider the mass and momentum equations in the liquid film with slip effects in the presence of the magnetic field simplified under the thin-film lubrication approximation. Considering \( (u, v, w) \) to be the Cartesian components of the velocity field in the liquid film, they are expressed as [27]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4}
\]

\[
\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} + \frac{B_0}{\mu_0} \frac{\partial b_x}{\partial z} = 0, \tag{5a}
\]

\[
\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} + \frac{B_0}{\mu_0} \frac{\partial b_y}{\partial z} = 0, \tag{5b}
\]

where \((b_x,b_y,0)\) are the components of the induced magnetic field in the presence of an external magnetic field \( B_0 \). \( \mu_0 \) is the magnetic permeability of free space and \( \mu \) is the dynamic viscosity. The Lorentz force including the applied and induced magnetic fields is represented as the last terms of Eqs. (5). The boundary condition for the components of the velocity field under the slip conditions at the heater plate are:

\[
u = U + \lambda \left( \frac{\partial u}{\partial z} \right)_{z=0} \quad \text{and} \quad w = 0 \text{ at } z = 0 \tag{6}
\]

\[
u = 0, \quad v = 0 \text{ and } w = \left( \frac{\rho_s}{\rho_l} \frac{dV}{dt} \right)_{z=\delta} \tag{7}
\]

Here, \( \lambda \) is the slip length for the velocity field. The slip effect is represented on the velocity component \((u, v)\) using the so-called Navier boundary condition. The mass balance at the interface between the liquid film and the solid arising from density differences between the phases is expressed in Eq. (7) [12].

2.1.2. Magnetic Induction Equations

By taking the curl of the Ohm’s law in the Maxwell’s equations, one obtains the following magnetic induction component equations [27]

\[
\frac{B_0}{\mu_0} \frac{\partial u}{\partial z} + \frac{1}{\mu_0} \frac{\partial^2 b_x}{\partial z^2} = 0, \tag{8a}
\]

\[
\frac{B_0}{\mu_0} \frac{\partial v}{\partial z} + \frac{1}{\mu_0} \frac{\partial^2 b_y}{\partial z^2} = 0. \tag{8b}
\]

First, the normal force balance is applied on the solid block. That is, the pressure in the liquid film \( p(x,y) \) is related to the weight and inertia of the solid block by

\[
M \left( g - \frac{dV}{dt} \right) = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{0}^{\frac{1}{2}} p(x,y) \, dx \, dy. \tag{1}
\]
The general boundary conditions for the magnetic induction equation taking into account the electrical conductivities of the liquid film, bottom plate and the solid block given by $\sigma$, $\sigma_p$, and $\sigma_s$ respectively, are represented as [27]

$$\frac{\partial b_x}{\partial z} - \frac{b_x}{(\chi_p, \delta)} = 0, \quad \frac{\partial b_y}{\partial z} - \frac{b_y}{(\chi_p, \delta)} = 0$$ at $z = 0 \quad (9)$

$$\frac{\partial b_x}{\partial z} + \frac{b_x}{(\chi_s, \delta)} = 0, \quad \frac{\partial b_y}{\partial z} + \frac{b_y}{(\chi_s, \delta)} = 0$$ at $z = \delta \quad (10)$

where

$$\chi_p = \frac{\sigma_p t_p}{\sigma \delta}, \quad \chi_s = \frac{\sigma_s H}{\sigma \delta}$$

Here, $\chi_p$ is the heated plate conductance ratio and $\chi_s$ is the solid block conductance ratio, and $H$ and $t_p$ are the height of the solid block and the heated plate thickness, respectively.

2.1.3. Energy Equation

The main mode of energy transport is by heat conduction, which is mediated by Joule heating arising from the induced current in the presence of magnetic field. The current density $j'$ can be obtained from the Ampere's law as [27]

$$j' = \frac{1}{\mu_0} \vec{v} \times \vec{B}, \quad (11)$$

where $\vec{B} = B_0 \vec{k} + \vec{b}$ and $\vec{b}$ is the induced magnetic field vector, with components $(b_x, b_y, 0)$.

Then, we can write the energy equation as

$$k \left( \frac{\partial^2 T}{\partial z^2} + \left( \frac{\partial b_x}{\partial z} \right)^2 + \left( \frac{\partial b_y}{\partial z} \right)^2 \right) = 0. \quad (12)$$

This equation (Eq. (12)) is subjected two different types of boundary conditions, viz., the constant temperature and constant heat flux, where the slip effect is applicable only for the former case. They are expressed as

a- Constant temperature heating

$$T = T_w + \lambda_H \left( \frac{\partial T}{\partial z} \right)$$ at $z = 0 \quad (13a) \quad T = T_m$ at $z = \delta \quad (13b)$

The second term of Eq. (13a) represents the temperature slip effect, where, $\lambda_H$ is the slip length for temperature.

b- Constant heat flux heating

$$-k \frac{\partial T}{\partial z} = q_w \text{ at } z = 0 \quad (14a)$$

$$T = T_m \text{ at } z = \delta \quad (14b)$$

2.2. Non-Dimensionalization

To simplify the solution of the velocity field, induced magnetic field and temperature, $(u, v, (b_x, b_y)$ and $T$, respectively, it is useful to non-dimensionalize the various conservation equations in the last section. Furthermore, we define the Hartmann numbers $H_a$ and $H$ according to the different characteristic length scales present in the problem of interest, i.e., the liquid film thickness $\delta$ and the solid block height $H$, as follows:

$$H_a = B_0 \delta \left( \frac{\sigma}{\mu} \right)^{\frac{1}{2}} \quad (15a)$$

and

$$H = H_a \mu \left( \frac{\sigma}{\mu} \right)^{\frac{1}{2}} \quad (15b)$$

These two dimensionless quantities are essentially related by

$$H_a = \frac{H}{H} \quad (15c)$$

Notice that during the transient close-contact melting process, $H_a$ varies as it is based on $\delta$, whereas $H$ remains an invariant. Furthermore, we define the following non-dimensional spatial coordinate, velocity fields and plate induced magnetic fields and the temperature field:

$$z^* = \frac{z}{\delta}, \quad u^* = \frac{u}{u_0}, \quad v^* = \frac{v}{v_0}, \quad U^* = \frac{U}{U_0}, \quad \frac{\partial v}{\partial z^*} \quad (16a)$$

and

$$b_x^* = \frac{b_x}{b_0}, \quad b_y^* = \frac{b_y}{b_0}, \quad \frac{\partial b_x^*}{\partial z^*} = -1, \quad (16b)$$

which are subjected to the following boundary conditions:

$$u^* = U^* + \frac{\lambda}{\delta} \frac{\partial u^*}{\partial z^*} \text{ at } z^* = 0 \quad (17a)$$

and

$$v^* = 0 \quad (17b)$$

$$v^* = 0 \text{ at } z^* = 1 \quad (18a)$$

Likewise, the magnetic induction equations (Eq. (8)), after non-dimensionalizing them become

$$\frac{\partial^2 b_x^*}{\partial z^2} + H_a \frac{\partial b_x^*}{\partial z^*} = 0 \quad (19a)$$

and

$$\frac{\partial^2 b_y^*}{\partial z^2} + H_a \frac{\partial b_y^*}{\partial z^*} = 0 \quad (19b)$$

and are subjected to the following boundary conditions:

$$\frac{\partial b_x^*}{\partial z^*} - \frac{b_x^*}{\chi_p} = 0, \quad \frac{\partial b_y^*}{\partial z^*} - \frac{b_y^*}{\chi_p} = 0 \text{ at } z^* = 0 \quad (20)$$

and

$$\frac{\partial b_x^*}{\partial z^*} - \frac{b_x^*}{\chi_s} = 0, \quad \frac{\partial b_y^*}{\partial z^*} - \frac{b_y^*}{\chi_s} = 0 \text{ at } z^* = 1 \quad (21)$$

The energy equation (Eq. (12)) can be non-dimensionalized, which, after simplification, can be written as

$$\frac{\delta^4}{\mu(T_w - T_m)} \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\delta^4}{\mu(T_w - T_m)} \left[ \frac{\partial^2 T}{\partial y^2} \right] \right] = 0. \quad (22)$$

For the boundary conditions of Eq. (22), we have the following results from Eqs. (13) and (14):

- Constant wall temperature $T_w$ heating mode (Case a):

$$\theta(z^* = 0) = 1 + \lambda_H \frac{\partial \theta}{\partial z^*} \text{ at } z^* = 0 \quad (23a)$$

and

$$\theta(z^* = 1) = 0 \quad (23b)$$
• Constant wall heat flux \( q_w \) heating mode (Case b):

\[
\left. \frac{\partial \theta}{\partial z^2} \right|_{z^*=0} = \frac{q_w \delta}{k(1 + \frac{1}{T_m})^2}, \quad \text{and} \quad \theta (z^* = 1) = 0
\] (24)

3. Analytical Solutions: Hydrodynamic Fields, Induced Magnetic Fields and Temperature Field in Melt Film

3.1. Solution of Velocity and Induced Magnetic Fields in the Liquid Film

The velocity and induced magnetic fields can be obtained by taking the first derivative Eq. (19a) with respect to \( z^* \) and then substituting it in Eq. (16a). This yields

\[
\frac{\partial^2 b_z^*}{\partial z^2} - \frac{\partial b_z^*}{\partial z^*} = \frac{H_d}{\Delta}
\]

Solving the above equation, we get the profile of the induced magnetic field in the \( x \)-direction \( b_z^* (z^*) \) as

\[
\psi_p = \left( \frac{H_d (\cosh (H_d) - 1)}{2 H_d} + \frac{2 \lambda H_d (\cosh (H_d) - 2 (\lambda + \delta) \sinh (H_d))}{2 H_d (\lambda + \delta)} \right) H_d
\]

\[
b_z^* (z^*) = K_1 \sinh (H_d z^*) + K_2 \cosh (H_d z^*) - \frac{z^*}{H_d}.
\] (25)

Here, the undetermined constants \( K_1 \) and \( K_2 \) can be obtained by using the boundary conditions given in Eqs. (20) and (21). We can then find the spatial variation of the velocity component in the \( x \)-direction, \( u^* (z^*) \) as

\[
u^* (z^*) = -K_1 \cosh (H_d z^*) - K_2 \sinh (H_d z^*) + K_3 z^* + K_4.
\] (26)

where \( K_3 \) and \( K_4 \) are constants and can be obtained by using the boundary conditions given in Eqs. (17) and (18). Similarly, the solution of the other components of the induced magnetic field in the \( y \)-direction, \( b_y^* (z^*) \), is obtained from Eq. (19b) subjected to Eqs. (20) and (21) and \( \psi^* (z^*) \) using Eq. (16b) subjected to the boundary conditions in Eqs. (17) and (18). Thus, we get

\[
b_y^* (z^*) = K_1 \sinh (H_d z^*) + K_2 \cosh (H_d z^*) - \frac{z^*}{H_d},
\] (27)

\[
u^* (z^*) = -K_1 \cosh (H_d z^*) - K_2 \sinh (H_d z^*) + K_3 z^* + K_4.
\] (28)

Here, the new constants \( K'_1 \) and \( K'_2 \) can be readily obtained from \( K_1 \) and \( K_2 \).

Then, we can obtain the dimensional volume flow rate in the liquid film in the \( x \)-direction, \( Q_x \), and in the \( y \)-direction, \( Q_y \), respectively, from the respective velocity fields as follows

\[
Q_x = \int_0^\delta u dz = \frac{\delta^3}{\mu} \left[ -\frac{K_1}{H_d} \sinh (H_d) \cosh (H_d z^*) \right. \\
\times \left. - \frac{K_2}{H_d} (\sinh (H_d) - 1) + \frac{K_3}{2} + K_4 \right],
\] (29)

\[
Q_y = \int_0^\delta v dz = \frac{\delta^3}{\mu} \left[ -\frac{K_1}{H_d} \sinh (H_d) \cosh (H_d z^*) \right. \\
\times \left. - \frac{K_2}{H_d} (\sinh (H_d) - 1) + \frac{K_3}{2} + K_4 \right].
\] (30)

Integrating the continuity equation Eq. (4)) across the liquid film thickness in the \( z \)-direction and then substituting Eqs. (29) and (30) into the resulting equation yields

\[
\frac{\partial Q_x}{\partial x} - \left( \frac{\rho_v V}{\rho} + \frac{(\rho - \rho_v) \delta}{\rho} \frac{d\delta}{dt} \right) \frac{\partial Q_x}{\partial y} = 0.
\]

It follows from the boundary conditions that \( K'_1 = K'_2 = K_1 \cosh (H_d - 1) + K_2 \sinh (H_d) \), \( K_3 = K_4 = K_1 \). Then using Eqs. (29) and (30) in the above integrated continuity equation, we get

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \left( \frac{\rho_v V}{\rho} + \frac{(\rho - \rho_v) \delta}{\rho} \frac{d\delta}{dt} \right)
\]

For ease of presentation in the following, we collect the group of terms within the square brackets in the left side of the previous equation and define a new factor \( \psi_p \) as

\[
\psi_p = \left( \frac{H_d (\cosh (H_d) - 1)}{2 H_d} + \frac{2 \lambda H_d (\cosh (H_d) - 2 (\lambda + \delta) \sinh (H_d))}{2 H_d (\lambda + \delta)} \right) H_d,
\] (31)

where \( \psi_p \) represents the effect of the electro-magnetic interactions on the close-contact melting problem in the presence of slip velocity effects, which is one of the first new analytical results arising from this analysis. Thus, influences the pressure field, which in turn modulates the velocity field within the liquid melt layer and hence the overall contact melting process. Therefore, the pressure field \( P \) satisfies

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = S,
\] (32)

which is a form of the Poisson equation, where the source term \( S \) in Eq. (32) is given by

\[
S = -\frac{1}{\psi_p} \left( \frac{\rho_v V}{\rho} + \frac{(\rho - \rho_v) \delta}{\rho} \frac{d\delta}{dt} \right).
\] (33)

Thus, Eq. (32) together with Eq. (33) is parameterized to include the Hartmann number \( H_d \) or \( H_a \) (magnetic field effects) and \( \lambda \) (velocity slip length effects) via \( \psi_p \) (Eq. (31)) on the hydrodynamics of the liquid film generated by contact melting. For completeness, the explicit expressions for the various integration constants \( K_1, K_2, K_3, K_4 \) appearing in the above, upon substituting the appropriate boundary conditions, can be written as

\[
K_1 = \frac{1}{H_d} \left[ \frac{x_{p1} H_d \sinh (H_d) + x_{p2} \cosh (H_d) + x_1 + 1}{(x_{p1} + x_1) H_d \cosh (H_d) + (x_{p2} H_d^2 + 1) \sinh (H_d)} \right],
\]

\[
K_2 = \frac{-x_{p1} H_d \cosh (H_d) + \sinh (H_d) - (x_1 + 1) H_d}{(x_{p1} + x_1) H_d \cosh (H_d) + (x_{p2} H_d^2 + 1) \sinh (H_d)}
\]

\[
K_3 = \frac{K_1 \cosh (H_d) - 1 + K_2 \sinh (H_d) + K_2}{2} H_d^2 - U^*
\]

\[
K_4 = U^* + K_1 + \frac{\lambda}{\rho} K_3 + \frac{\lambda}{\rho} H_d K_2.
\]

3.2. Solution of Poisson Equation for Pressure

The pressure field \( P(x, y) \) (i.e., the Poisson equation) in the liquid film in the Eq. (32) can be solved with specific boundary conditions. The liquid film exposed to the outside environment can be
presumed to be in equilibrium with it [7]. Therefore, it follows that \( P(x = \pm \frac{1}{2}, y) = 0 \) and \( P(x, y = \pm \frac{1}{2}) = 0 \). The boundary conditions of the Poisson equation are uniform and therefore it can be readily solved analytically by using a variable separable method involving a Fourier series, specifically using the principle of superposition, as in [7]

\[
P(x, y) = \sum_{n=0}^{\infty} C_n \cos \left( \beta_n x \right) \cosh \left( \beta_n y \right).
\]

Here, \( \beta_n \) are the eigenvalues of the Fourier series given by \( \beta_n = (2n+1) \frac{\pi}{2} \), \( n = 0, 1, 2, \ldots \) and \( C_n \) is the Fourier coefficient, which is obtained as \( C_n = \frac{2\epsilon}{\beta_n \cosh(\beta_n L/2)} \). We then get the final form of the normal force balance by substituting for \( P(x, y) \) from in Eq. (1), which can be written as

\[
\rho_L \omega L W H G = \left( \frac{\mu U L^3}{12 \varphi_p \beta^2} \right) \left( \frac{p_L V}{\rho_L} + \frac{(p_L - p_R)}{\rho_L} \frac{d\delta}{dt} \right) G'(A).
\]

Here, \( G'(A) \) is given by \( G'(A) = 1 - \frac{1}{2} \frac{\sigma^2 U^2 \delta^2}{\rho_L T} \), where \( A = \frac{W}{L} \). The additional pre-factor \( 1/(12\varphi_p) \) in Eq. (34) represents the influence of the electromagnetic field and slip effects (via Eq. (31)) when compared to the solution presented in [7].

3.3. Solution of Temperature Field

A scale analysis shows that the Joule heating effects can be neglected in Eq. (12), if

\[
\sigma B^2 U^2 \delta^2 \frac{d^2 T}{k dT} \ll 1.
\]

Assuming that the imposed magnetic field \( B_0 \), temperature difference (\( \Delta T = T_w - T_m \)), heater sliding velocity \( U \) and the properties of the fluid are such that they satisfy this constraint (see the appendix for dealing with the more general case), the energy equation (Eq. (12)) then simplifies to

\[
k \frac{d^2 T}{dz^2} = 0,
\]

which in non-dimensional form obtained becomes

\[
\frac{d^2 \theta}{dz^2} = 0.
\]

By integrating Eq. (36) twice, we get

\[
\theta = K_2 z^* + K_3.
\]

The new constants \( K_2 \) and \( K_3 \) subjected to the boundary conditions are specified in what follows:

(a) Case of constant wall temperature

Applying the boundary conditions presented in Eq. (23) in the above equation, we get

\[
\theta = (1 - z^*) \frac{1}{1 + \frac{2H}{\beta}}.
\]

and

\[
\frac{d\theta}{dz} = -\frac{1}{1 + \frac{2H}{\beta}}.
\]

(b) Case of constant wall heat flux

Also, by applying the boundary conditions in Eq. (24) we obtain

\[
\theta = \frac{q_w}{k(T_w - T_m)} (1 - z^*),
\]

and

\[
\frac{d\theta}{dz} = \frac{-q_w}{k(T_w - T_m)}.
\]

3.4. Energy Balance at the solid-liquid interface

Then, the energy balance equation at the solid-liquid interface given by Eq. (2) reads as

\[
f \left( \frac{k(T_w - T_m)}{\delta} \right) \frac{d\theta}{dz} \bigg|_{z=1} = \rho_L h_f \left( V + \frac{\partial \delta}{\partial t} \right).
\]

For the constant temperature wall heating, i.e., case (a), this equation reduces to

\[
f \left( \frac{k(T_w - T_m)}{\delta} \right) \frac{1}{1 + \frac{2H}{\beta}} = \rho_L h_f \left( V + \frac{\partial \delta}{\partial t} \right).
\]

which accounts for the temperature slip effect via \( \lambda_{sl} \). Similarly, for the constant heat flux wall heating, i.e., case (b), we have

\[
q_w = \rho_L h_f \left( V + \frac{\partial \delta}{\partial t} \right).
\]

3.5. Tangential Force and Coefficient of Friction Calculation

Finally, the tangential force is obtained from Eq. (3). To obtain an explicit expression, we first non-dimensionalize the velocity gradient in Eq. (3) as

\[
F_x = -\frac{\delta^2}{\Omega^2} \left( \frac{d\mu}{dx} \right) \left( \frac{1}{2} \right) \int \left[ \int \mu \frac{\partial u^*}{\partial z} |_{z=0} dx \right] dy,
\]

which simplifies to

\[
F_x = \frac{\mu ULW}{\delta + \lambda}.
\]

Thus, the coefficient of friction \( \mu_f \) can be obtained from Eq. (42) as follows:

\[
\mu_f = \frac{F_x}{Mg} = \frac{\mu ULW}{Mg(\delta + \lambda)},
\]

which is influenced by the velocity slip effects via the appearance of \( \lambda \).

3.6. Further Non-dimensionalization of the Quantities of Interest

To decrease the number of parameters and in order to show the characteristic dimensionless groups for the contact-melting problem in the presence of magnetic field involving both slip velocity and slip temperature effects, we now introduce a non-dimensionalization of the above equations. Hence Eq. (34) can be rewritten in terms of the dimensionless quantities previously defined in [13] as

\[
\frac{\delta^3}{\mu H} (12\varphi_p) = \left( \frac{L}{R} \right)^2 G'(A) \left( \bar{V} + (1 - \bar{\rho}) \frac{\partial \delta}{\partial t} \right).
\]

As in [13], we adopt \( (UL)^3/2 \), which is the geometric mean of the length scale of the contact area, as our characteristic length. This is suitable for obtaining the influence of the aspect ratio when the contact area is kept invariant. Thus, the non-dimensional normal force balance (Eq. (34)) can be rewritten as

\[
\bar{V} + (1 - \bar{\rho}) \frac{\partial \delta}{\partial t} = \left( \frac{\delta^3}{\mu H} G'(A) \right) (12\varphi_p).
\]

Now, from Eq. (15a) and Eq. (15c), the Hartmann numbers based on the scales of \( \delta \) and \( H \) are related by \( Ha_q = Ha(\frac{H}{R}) \). Here, \( Ha \) is a constant obtained from the statement of the system and is a constant. Additionally, the above equation contains the factor \( \varphi_p \), which needs to be rewritten in dimensionless form. Recalling that
\(K_1\) and \(K_2\) are given in Eq. (31) and substituting for the dimensionless quantities, and considering the above relation between the Hartmann numbers, we get

\[
\varphi_p = \left( \frac{\tilde{\delta}(Ha_H^2)}{2(\lambda + \delta)} \right) K_1 + \left( \frac{\sinh(Ha_H^2)}{2(\lambda + \delta)} \right) K_2
\]

(45)

Equations (39) and (40) become independent of the length scale as follows. First, considering the non-dimensional version of Eq. (39), i.e.,

\[
\tilde{V} + \frac{\tilde{\delta}}{\tilde{\tau}} = \tilde{\rho} \cdot \text{St} = \frac{\tilde{\rho} \cdot \text{Ste}}{\tilde{\lambda} H_{\text{m}}}
\]

(46)

and since the non-dimensional energy balance for case (a) (Eq. (39)) finally becomes

\[
\tilde{V} + \frac{\tilde{\delta}}{\tilde{\tau}} = \tilde{\rho} \cdot \text{Ste} = \frac{\tilde{\rho} \cdot \text{Ste}}{\tilde{\lambda} H_{\text{m}}}
\]

Similarly, considering the non-dimensional form of Eq. (39), i.e.,

\[
\tilde{V} + \frac{\tilde{\delta}}{\tilde{\tau}} = \tilde{\rho} \cdot \tilde{q}_w = \tilde{\rho} \cdot \frac{\tilde{q}_w}{\tilde{\rho}_s \tilde{h}_{\text{ref}}}
\]

Since the non-dimensional wall heat flux \(\tilde{q}_w\) is equal to \(\frac{\tilde{\rho} \cdot \text{Ste}}{\tilde{\lambda} H_{\text{m}}}\), the non-dimensional energy balance for case (b) (Eq. (40)) finally reads as

\[
\tilde{V} + \frac{\tilde{\delta}}{\tilde{\tau}} = \tilde{\rho} \cdot \tilde{q}_w
\]

(47)

4. Asymptotic Analytical Solutions under Limiting Cases

4.1. No Magnetic Field

For electrically insulating materials (\(\chi_p \text{ and } \chi_s = 0\)) and in the absence of magnetic field (\(Ha_s = 0\)), we can use Taylor series for the hyperbolic functions in the \(\varphi_p\) term that appears in the source term \(S\) in Eq. (33), where the factor \(\varphi_p\) is defined by Eqs. (45) for further simplification:

\[
\varphi_p = \left( \frac{\tilde{\delta} Ha_s (\cosh(Ha_s) + 1) + 2\lambda Ha_s (\cosh(Ha_s) - 2(\lambda + \delta) \sinh(Ha_s))}{2Ha_s (\lambda + \delta)} \right) \frac{1}{Ha_s} \frac{1}{\sinh(Ha_s)}
\]

Then, by setting \(Ha_s \rightarrow 0\), the factor \(\varphi_p\) reduces to the following:

\[
\lim_{Ha_s \rightarrow 0} \varphi_p = \frac{1}{12} \left( \frac{4\lambda + \delta}{\lambda + \delta} \right)
\]

(48)

Thus, when \(\varphi_p = \frac{1}{12} \left( \frac{4\lambda + \delta}{\lambda + \delta} \right)\), the formulation corresponds to the non-magnetic case, but with slip effects and which also further recovers the previous analytical results without slip effect, i.e., \(\varphi_p = 1/12\) when \(\lambda = 0\) [7,13]. The slip effects are represented in the term \(\frac{4\lambda + \delta}{\lambda + \delta}\), which is another new analytical result arising from this study.

4.2. Small Hartmann Number \(Ha_s\)

In this case, let us consider contact-melting in the presence of magnetic field with small Hartmann numbers i.e., \(Ha_s \ll 1\) or \(Ha_s (\delta) \ll 1\) but \(Ha_s \neq 0\) and for any \(\chi_s\) and \(\chi_p \neq 0\). Now, we can apply linearization in the Taylor series expansion of the hyperbolic functions by retaining terms up to the order of \(Ha_s^2\) in the above formulation. Then, Eq. (45) for the \(\varphi_p\) term representing the electro-magnetic effects with slip effects at relatively small Hartmann numbers simplifies to

\[
\varphi_p \approx \frac{Ha_s^2}{12} \left( \frac{4\lambda + \delta}{\lambda + \delta} \right) K_1 + \left( \frac{Ha_s^3}{24} + \frac{Ha_s \lambda (2\lambda + \delta) - Ha_s \lambda}{\lambda + \delta} \right) K_2
\]

(49)

Substituting for the integration constants \(K_1\) and \(K_2\), Eq. (50) modifies to

\[
\varphi_p \approx \left( \frac{4\lambda + \delta}{\lambda + \delta} \right) \frac{1}{12} \left[ \left( \frac{2\lambda + \delta}{\lambda + \delta} - \frac{24\lambda}{\delta} \right) + \left( \frac{1}{2} \frac{4\lambda + \delta}{\lambda + \delta} - \frac{6\lambda (2\lambda + \delta)}{(\lambda + \delta)} - \frac{24\lambda}{\delta} \right) \right]
\]

(51)

Now, by inspecting Eq. (51), we can define a new effective electrical conductance factor \(\omega\) given by

\[
\omega = \frac{4\lambda + \delta}{\lambda + \delta} \frac{1}{12} + \frac{1}{2} \left( \frac{4\lambda + \delta}{\lambda + \delta} - \frac{6\lambda (2\lambda + \delta)}{(\lambda + \delta)} - \frac{24\lambda}{\delta} \right)
\]

(52)

which indicates the net effect of the electrical conductance \(\chi_s\) and \(\chi_p\) on the contact-melting system in the small \(Ha_s\) limit. Hence, we finally get

\[
\varphi_p = \left( \frac{4\lambda + \delta}{\lambda + \delta} - \frac{6\lambda (2\lambda + \delta)}{(\lambda + \delta)} - \frac{24\lambda}{\delta} \right) \left[ 1 + \omega Ha_s \frac{\delta^2}{H^2} \right]
\]

(53)

The terms within the parentheses (...) in Eq. (53) represents the slip effects on the contact-melting problem, while the \(\omega\) term in the same equation and factor \(Ha_s\) indicates the correction or the linearized effect due to the magnetic field for small \(Ha_s\). Hence, in such situation, the non-dimensional normal force balance (Eq. (44)) further reduces to

\[
\tilde{V} + (1 - \tilde{\rho}) \frac{\tilde{\delta}}{\tilde{\tau}} = \tilde{\rho}_g \frac{\tilde{G}_{\text{Pr}}}{\tilde{G}_{\text{Pr}}} \left( \frac{4\lambda + \delta}{\lambda + \delta} - \frac{6\lambda (2\lambda + \delta)}{(\lambda + \delta)} - \frac{24\lambda}{\delta} \right) \left[ 1 + \omega Ha_s \frac{\delta^2}{H^2} \right]
\]

(54)

5. Results and Discussion

A systematic investigation is now performed to study the transient and steady state contact-melting process subjected to a magnetic field with slip effects in velocity and/or temperature. The electromagnetic effects are characterized by the Hartmann number and the conductance ratio of the heated plate and the solid block. On the other hand, the slip effects on velocity and temperature are described by \(\lambda\) and \(\lambda_H\), respectively. The reference conditions for the baseline parameters used for the computations in this study are presented in Table 1. Equations (44) together with (45) and either Eq. (46) or (47) are solved numerically to obtain the melting rate \(\tilde{V}(t)\) and the melt layer thickness \(\tilde{\delta}(t)\).
Fig. 2. Comparison of present numerical results with previous reference results [13] for the constant wall temperature heating mode in the limiting case for (a) normalized melt velocity, and (b) normalized film liquid thickness.

Table 1
Reference conditions used for computations in this study.

<table>
<thead>
<tr>
<th>Heating mode</th>
<th>Constant wall temperature</th>
<th>Constant heat flux</th>
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</thead>
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<tr>
<td>Cross-sectional shape</td>
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<td>Axisymmetric</td>
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<tr>
<td></td>
<td>[10,11]</td>
<td>[10,13]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
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<td>13.44</td>
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<td>1.0</td>
</tr>
<tr>
<td>g</td>
<td>$5.521 \times 10^{11}$</td>
<td>$5.521 \times 10^{11}$</td>
</tr>
<tr>
<td>G</td>
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</tr>
<tr>
<td>Ste</td>
<td>1.266 $\times 10^{-2}$</td>
<td>--</td>
</tr>
<tr>
<td>$\tilde{\phi}_W$</td>
<td>--</td>
<td>24.57</td>
</tr>
</tbody>
</table>

5.1. Validation

The analytical model was first validated with previous reference study reported in Ref. [13] in the limiting case. This corresponds to the case of setting the slip lengths in velocity and temperature to zero and considering no imposed magnetic field in our model. Fig. 2 shows comparisons of the numerical results based on our analytical solution representing the evolution of the normalized melting velocity and liquid film thickness for the constant wall temperature mode. Here, as in Ref. [13], the normalized melting velocity $\tilde{V}$, normalized liquid film thickness $\tilde{\delta}$ and normalized time $\tilde{\tau}$ are obtained as follows: $\tilde{V} = \frac{V}{\tilde{V}_c}$, $\tilde{\delta} = \frac{\delta}{\tilde{\delta}_c}$ and $\tilde{\tau} = \frac{\tau}{\tilde{\tau}_c}$, where $\tilde{V}_c$, $\tilde{\delta}_c$ and $\tilde{\tau}_c$ are the dimensionless steady state values of the melt velocity and film liquid thickness, respectively. The computed results based on our analytical model are found be in excellent agreement with the reference results.

5.2. Effect of Slip Lengths with small and high values of Hartmann Number

For constant wall temperature heating mode, Figs. (3a) and (3b) show the effect of the slip length for velocity $\lambda$ in the absence of temperature slip effect ($\lambda_H = 0$) and small Hartmann number ($Ha = 1 \times 10^{-3}$). It is seen that the normalized solid descending velocity, $\tilde{V}$, increases steadily with increasing the slip length for velocity, $\lambda$. On the other hand, the time to reach steady state is found to decrease slightly as the velocity slip length $\lambda$ increases. From Fig. (3b), the opposite effect is realized for the normalized liquid film thickness, $\tilde{\delta}$, which drops as the velocity slip length increases $\lambda$. Similarly, the time to reach steady state is increased as the velocity slip length, $\lambda$, decreases. When temperature slip length $\lambda_H$ is taken equal to the slip length for velocity i.e., $\lambda_H = \lambda$, and at small Hartmann number $Ha$, from Fig. (4a), it is noticed that the normalized melt velocity, $\tilde{V}$, decreased steadily with increasing the respective slip lengths.

On the other hand, in Fig. (4b), the normalized liquid film thickness, $\tilde{\delta}$, decreases as the slip lengths for velocity and temperature slip are increased. For both melt velocity, $\tilde{V}$ and normalized liquid film thickness, $\tilde{\delta}$, the attainment of steady state is somewhat delayed with increase in slip lengths as can be seen from Fig. (4a) and (4b). At a high value of the Hartman number $Ha = 10000$, Figs. (5a) and (5b) show that the normalized solid descending velocity, $\tilde{V}$, increases as slip length for velocity, $\lambda$, increased in the absence of the slip temperature effect. On the other hand, the normalized liquid film thickness, $\tilde{\delta}$, decreases as the velocity slip length gradually increases (see Fig. (5b)). When compared to the result for the no magnetic field case (see Figs. (3a) and (3b)), we see that an increase in $Ha$ for similar slip effects leads to slower melting rate in thicker melt layer.

Figs. (6a) and (6b) illustrate the effect of slip in temperature, where the slip length for temperature is set equal to the slip velocity at a higher Hartmann number $Ha = 10000$. From Fig. 6a, it is observed that the normalized melt velocity, $\tilde{V}$, decreases progressively with increasing slip lengths. In Fig. (6b), the liquid is seen to become thinner as the slip lengths for velocity and temperature increase. For both the melting rate $\tilde{V}$ and the thickness $\tilde{\delta}$, the time to reach steady state is found to be delayed (see Figs. (6a) and (6b)).

For constant wall heat flux heating mode, the slip effects on the velocity and with or without those on the temperature for the constant wall heat flux case on the normalized solid descending velocity, $\tilde{V}$, as function of time are presented in Figs. (7a) and (8a), respectively. The result demonstrate that the slip does not affect the melting rate, $\tilde{V}$, for both small and high $Ha$. Furthermore, the greater magnetic field strength, i.e., $Ha = 10000$, the longer it takes to melt the solid block, which is similar to the previous case. On the other hand, correspondingly, the normalized liquid film thickness, $\tilde{\delta}$, variation with time is also illustrated in Figs. (7b) and (8b). The presence of slip is seen to cause a significant decrease in the melt film thickness similar to the previous case. The effect of slip with a constant wall heat flux on the normalized liquid film thickness, $\tilde{\delta}$, results in faster attainment to its steady state values, as evidenced, from Figs. (7) and (8).

5.3. Effect of both conductance ratios of the solid block $\chi_S$ and the heated plate $\chi_P$

In Figs. (9a) and (9b) we consider conductance ratio $\chi_S$ equal to $\chi_P$ with high Hartman number $Ha = 10000$. For either the presence or the absence of the slip effect, the normalized melt velocity, $\tilde{V}$, is seen to increase when the values of both conductance
ratio of the solid block, $\chi_S$, and conductance ratio of the plate, $\chi_D$, are decreased. In contrast, the normalized liquid film thickness, $\tilde{\delta}$, increases as conductance ratios increase as illustrated in Figs. (9a) and (9b).

5.4. Effect of gravitational acceleration $\tilde{g}$

Figs. (10a) and (11a) illustrate the effect of gravitational acceleration $\tilde{g}$ on the normalized solid descending velocity $\tilde{V}$ subjected to magnetic field with both small and high Hartman number i.e., $Ha = 1 \times 10^{-3}$ and $Ha = 10000$, respectively, under the constant wall temperature heating mode in the presence of slip effect. It is found that increasing the gravitational acceleration results in significantly higher melting rates. The other main influence of increasing $\tilde{g}$ on the contact melting in the presence of magnetic field is to reduce the melt film thickness, as can be seen in Figs. (10b) and (11b). In addition, the contact melting process is seen to attain its steady state faster at higher $\tilde{g}$.

5.5. Effect of Prandtl number $Pr$

The effect of the Prandtl number, $Pr$, on the normalized solid descending velocity, $\tilde{V}$, in the presence of slip is shown in Figs. (12a) and (13a) at two different Hartmann numbers. As the Prandtl
number increases, it is seen that the melt rate decreases significantly for both small and high Hartman numbers i.e., \( Ha = 1 \times 10^{-3} \) and \( Ha = 10000 \). It is seen that lower Prandtl number liquids are expected to undergo faster melting when compared to that of higher \( Pr \) materials. Moreover, it is found that the time required to reach steady state progressively increases as \( Pr \) increases. On the other hand, Figs. (12b) and (13b) present the influence of \( Pr \) on the liquid film thickness, \( \delta \). It is obvious that an increase in \( Pr \) results in the thickening of the liquid film during contact melting for both small and high \( Ha = 1 \times 10^{-3} \) and \( Ha = 10000 \) under slip effects. The time needed to reach its steady state value is also found to increase as \( Pr \) increases. For both transient processes, it is seen that the slip effect slightly delay the attainment of steady state.

**Conclusions**

In this work, we have investigated the influence of slip effects on the contact melting heat transfer due to a relative motion between a phase change material (PCM), and a heated plate, both of them being electrically conducting, in the presence of a magnetic field. An explicit closed-form analytical solution was developed for unsteady close-contact melting under both isothermal and con-
constant heat flux wall heating modes. Numerical solution of our formulation under asymptotic limiting conditions for the melting rate and liquid film thickness are found to be in excellent agreement with previous reference results and provides a validation of this work. Results were obtained for both isothermal and constant heat flux wall heating modes for a set of Hartmann number $Ha$, Prandtl number $Pr$, plate conductance ratio $\chi_p$, solid block conductance ratio $\chi_s$, and dimensionless acceleration $\bar{g}$. The following are the main findings of this study:

- Increasing the velocity slip length $\bar{\lambda}$, when slip length for temperature $\lambda_H = 0$, is found to result in a faster melting rate; correspondingly, the melt layer becomes thinner. Furthermore, it results in faster attainment of the steady state.
When both the slip length in velocity and temperature are increased by the same magnitude i.e., with \( \tilde{\lambda} = \tilde{\lambda}_H \), by contrast, it leads to slower melting rates. However, as before the liquid film gets thinner for any increase of the slip length. Remarkably, for such cases, the approach to steady state is delayed with when both \( \tilde{\lambda} \) and \( \tilde{\lambda}_H \) are increased by the same value.

When the Hartmann numbers \( H\alpha \) are increased for any choice of slip length, they lead to slower melting rates and are accompanied by thicker melt layers. However, when the conductance ratios of the heated plate \( \chi_p \) and solid \( \chi_s \) increased, behaviors that are opposite to the above are observed.

The main influence of increasing the gravitational acceleration \( \tilde{g} \) on contact melting by direct heating with slip effects in the presence of magnetic field is faster melting rates with corresponding thinning of the liquid film. The duration to approach steady states is shortened.

The effect of increasing Prandtl number \( Pr \) in the presence of slip at the heated plate causes slower melting; by contrast, the liquid film becomes thicker as \( Pr \) is increased.

The present work can be extended further in various ways in future studies. These include incorporating additional physical features and configurations in the modeling of microscale contact melting with slip effects and magnetic effects, such as rotational effects, permeable media, and nanoparticle additives. The resulting models are expected to be more complex for representing multi-physics attendant effects, which can be resolved by more refined numerical solution strategies. Furthermore, such investigations could be complemented by carrying out experimental work on contact melting processes with such additional features.

Declaration of Competing Interest

None.

CRediT authorship contribution statement

\textbf{Mutabe Aljaghtham}: Conceptualization, Investigation, Data curation, Software, Writing - original draft. \textbf{Kannan Premnath}: Conceptualization, Supervision, Writing - review & editing. \textbf{Radi Alsulami}: Investigation, Writing - review & editing.

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Appendix. Solution of temperature field with Joule heating

In the main section of this paper, we neglected the effects of Joule heating in the energy equation. In this appendix, the solution of temperature field in the presence of Joule heating is presented.
Equation (22) requires the pressure gradients. From the results presented in Sec. 3.2, it follows that

$$\frac{\partial P}{\partial x} = SL \left[ \frac{X}{L} \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi^2(2n+1)^2} \sin \left( 2n+1 \right) \pi \frac{X}{L} \frac{\cosh \left( 2n+1 \pi A_{\frac{X}{L}} \right)}{\cosh \left( 2n+1 \pi A \right)} \right],$$

(A-1)

and

$$\frac{\partial P}{\partial y} = SL \left[ \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi^2(2n+1)^2} \cos \left( 2n+1 \right) \pi \frac{X}{L} \frac{\sinh \left( 2n+1 \pi A_{\frac{X}{L}} \right)}{\cosh \left( 2n+1 \pi A \right)} \right].$$

(A-2)

Based on these two equations, for convenience, we define the following functions

$$\pi_x \equiv \pi_x \left( \frac{X}{L}, \frac{Y}{W} \right) = \frac{X}{L} \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi^2(2n+1)^2} \sin \left( 2n+1 \right) \pi \frac{X}{L} \frac{\cosh \left( 2n+1 \pi A_{\frac{X}{L}} \right)}{\cosh \left( 2n+1 \pi A \right)} .$$

(A-3)

$$\pi_y \equiv \pi_y \left( \frac{X}{L}, \frac{Y}{W} \right) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi^2(2n+1)^2} \cos \left( 2n+1 \right) \pi \frac{X}{L} \frac{\sinh \left( 2n+1 \pi A_{\frac{X}{L}} \right)}{\cosh \left( 2n+1 \pi A \right)} .$$

(A-4)

Hence, we can rewrite Eqs. (A-1) and (A-2) as follows:

$$\frac{\partial P}{\partial x} = SL \pi_x \left( \frac{X}{L}, \frac{Y}{W} \right), \frac{\partial P}{\partial y} = SL \pi_y \left( \frac{X}{L}, \frac{Y}{W} \right).$$

(A-5)

Thus, the non-dimensional equation for the temperature field (Eq. (22)) becomes

$$\frac{\partial^2 \theta}{\partial z^2} + \frac{\delta^4}{k \mu (T_m - T_m)} S^2 L^2 \left[ \pi_z \left( \frac{\partial b_x}{\partial z} \right)^2 + \pi_z \left( \frac{\partial b_y}{\partial z} \right)^2 \right] = 0 .$$

(A-6)

Now, from Eqs. (25) and (27), we get

$$\frac{\partial b_x}{\partial z} = \frac{\partial b_y}{\partial z} = K_1 H a \cosh (H a z) + K_2 H a \sinh (H a z) - \frac{1}{H a \delta} .$$

(A-7)

Substituting Eq. (A-7) in (A-6), we get

$$\frac{\partial^2 \theta}{\partial z^2} + \frac{\delta^4}{k \mu (T_m - T_m)} S^2 L^2 \left( \pi_z^2 + \pi_z^2 \right) H a^2 \left[ \frac{K_1 \cosh (H a z) + K_2 \sinh (H a z)}{H a \delta} \right]^2 = 0 .$$

In order to simplify this last equation, we define a new factor $\Gamma$ as

$$\Gamma = \frac{\delta^4}{k \mu (T_m - T_m)} S^2 L^2 \left( \pi_z^2 + \pi_z^2 \right) H a^2 .$$

(A-8)

Substituting for the source term $S$ given in Eq. (33) in Eq. (A-9), it follows that

$$\Gamma \frac{\partial^2 \theta}{\partial z^2} + \Gamma \left[ K_1 \cosh (H a z) + K_2 \sinh (H a z) \right] = 0 .$$

(A-9)

Now, by integrating Eq. (A-9) twice, we get the following expression for the temperature profile $\theta$ with two undetermined constants $K_5$ and $K_6$:

$$\theta = -\Gamma \left[ \frac{1}{2 K_5} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right] - \frac{K_5}{H a \delta} \cosh (2 H a z) \frac{\delta}{k (T_m - T_m)} - \frac{K_6}{H a \delta} \sinh (2 H a z) \left( K_1^2 - K_2^2 + \frac{1}{H a \delta} \right) z^2 + K_5 z^2 + K_6 .$$

(A-10)

To obtain the expression for the constants $K_5$ and $K_6$, we now consider the following two different heating modes and the corresponding boundary conditions.

Case (a): constant wall temperature heating mode

Applying $\theta(z^* = 0) = 1$, we get

$$K_5 = 1 - \Gamma \left[ \frac{1}{8 H a \delta} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right] .$$

(A-11)

Similarly, using $\theta(z^* = 1) = 0$, yields

$$K_6 = 1 - \Gamma \left[ \frac{1}{2 H a \delta} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right] - K_5 .$$

(A-12)

Case (b): constant wall heat flux heating mode

Now, applying the heat flux boundary condition at $z^* = 0$, we get

$$K_5 = \Gamma \left[ \frac{K_1 K_2}{2 H a \delta} - \frac{2 K_2}{H a \delta} \right] - \frac{q_w \delta}{k (T_m - T_m)} .$$

(A-13)

In similar manner, setting $\theta(z^* = 1) = 0$ results in the following:

$$K_6 = \Gamma \left[ \frac{1}{4 H a \delta} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right] - \frac{2 K_2}{H a \delta} \sinh (2 H a z) + \left( K_1^2 - K_2^2 + \frac{1}{H a \delta} \right) z^2 - K_5 .$$

(A-14)

Now, the energy balance equation at the solid-liquid interface Eq. (2) is given by

$$- \frac{\partial (T_m - T_m)}{\delta} \left( \frac{\partial \theta}{\partial z} \right) |_{z^* = 1} = \rho_i h_i \left( V + \frac{d \rho}{d t} \right) .$$

(A-15)

Based on Eq. (A-13), the temperature gradient at the solid-liquid interface ($\frac{d \theta}{d z}$) at $z^* = 1$ can be rewritten as

$$\frac{\partial \theta}{\partial z} \bigg|_{z^* = 1} = -\Gamma \left[ \frac{1}{4 H a \delta} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right] - \frac{2 K_2}{H a \delta} \sinh (2 H a z) + \left( K_1^2 - K_2^2 + \frac{1}{H a \delta} \right) z^2 + K_5 .$$

(A-16)

Hence, the final form of the energy balance equation at the solid-liquid interface for the constant wall temperature heating mode (case (a)) becomes

$$\frac{1}{\rho_i h_i} \left( \frac{1}{4 H a \delta} \left( K_1^2 + K_2^2 \right) \cosh (2 H a z) + \frac{K_5}{H a \delta} \sinh (2 H a z) \right) - \frac{2 K_2}{H a \delta} \sinh (2 H a z) + \left( K_1^2 - K_2^2 + \frac{1}{H a \delta} \right) z^2 + K_5 = \rho_i h_i \left( V + \frac{d \rho}{d t} \right) .$$

(A-17)
Similarly, for the constant wall heat flux heating mode (case b), the interface energy balance equation transforms to

\[
\frac{1}{\rho C_p} \left( \frac{\partial}{\partial x} \right) \left( \rho C_p \left( \frac{\partial T}{\partial x} + \frac{V}{\rho g} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \right) - \frac{\partial \rho C_p h}{\partial t} \right) = \frac{1}{\Delta x^2} \left( K_2^2 + K_1^2 \sinh(2H_{a_1}) + \frac{K_2 K_0}{\Delta x^2} \cosh(2H_{a_1}) - \frac{2K_0}{\Delta x^2} \sinh(H_{a_1}) - \frac{K_0}{\Delta x^2} \cosh(H_{a_1}) \right) \left( K_1^2 - K_2^2 + \frac{1}{\Delta x^2} \right) \cdot \frac{1}{\Delta x^2} \left( \frac{\partial}{\partial x} \right) \left( \frac{\partial T}{\partial x} + \frac{V}{\rho g} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \right) \frac{1}{\Delta x^2} \left( K_2^2 + K_1^2 \sinh(2H_{a_1}) + \frac{K_2 K_0}{\Delta x^2} \cosh(2H_{a_1}) - \frac{2K_0}{\Delta x^2} \sinh(H_{a_1}) - \frac{K_0}{\Delta x^2} \cosh(H_{a_1}) \right) \cdot \frac{1}{\Delta x^2} \left( \frac{\partial}{\partial x} \right) \left( \frac{\partial T}{\partial x} + \frac{V}{\rho g} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \right)
\]

(A-18)

It may be noted that since the terms \( \pi_x \) and \( \pi_y \) are dependent on the spatial coordinates \( x/L \) and \( y/W \), it follows that the temperature profile \( \theta \), liquid film thickness \( \delta \) and the solid descending velocity \( V \) are all functions of the local horizontal spatial coordinates in addition to time, i.e., \( \theta(x/L, y/W, z/\delta, t) \), \( \delta(x/L, y/W, z/\delta, t) \) and \( V(x/L, y/W, z/\delta, t) \) when Joule heating effects are considered. In contrast, if the Joule heating is ignored as done in the main section of this paper, all the above quantities are independent of \( x/L \) and \( y/W \) and they just vary only in the vertical coordinate direction \( z \) normal to the interface, which considerably simplifies the model and the resulting analysis.

References