The Incomplete Pochhammer Symbols and Their Application to Generalized Hypergeometric Functions

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Abstract

In this present paper, we deal with the familiar incomplete gamma functions \( \gamma(s, x) \) and \( \Gamma(s, x) \) and the incomplete Pochhammer symbols, to introduce some integral, differentiation formulas of a class of generalized incomplete hypergeometric functions \( _2\Gamma_1(a,b;c;\tau,z) \) and \( _p\Gamma_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;\tau,z) \). Also, we establish the integral representation, some recurrence relations and Mellin Barnes integral of the generalized incomplete hypergeometric function \( _2\Gamma_1(a,b;c;\tau,z) \).

1 Introduction

The well known incomplete gamma functions \( \gamma(s, x) \) and \( \Gamma(s, x) \) be defined by

\[
\gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt, \quad R(s) > 0, x \geq 0
\]
and

\[
\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt, \quad R(s) > 0, x \geq 0,
\]
respectively, (1.1) and (1.2) satisfy the following decomposition formula

\[
\gamma(s, x) + \Gamma(s, x) = \Gamma(s), \quad R(s) > 0.
\]

Each of these function have a vital role in the study of the analytic solutions of a variety of problems in diverse area of science, probability and engineering. (see [1]-[3],[5],[8],[9],[12]-[14],[18]). In term of the gamma function \( \Gamma(s) \), the Pochhammer symbol \( (z)_n \) can be defined by

\[
(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \begin{cases} 1, & n = 0 \\ z(z+1)(z+2)\cdots(z+n-1), & n \in \mathbb{N}, z \in \mathbb{C}, \end{cases}
\]

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where \( \mathbb{N} \) denotes the set positive integers. In particular, the Pochhammer symbol \((z)_n\) satisfy the following relations (see\[14\])

\[
(z)_{m+n} = (z)_m(z + m)_n, \\
(z)_{-n} = \frac{(-1)^n}{(1 - z)_n}, \\
(2z)_{2n} = 2^{2n}(z)_n(z + \frac{1}{2})_n, \\
(z)_{n-k} = \frac{(-1)^k(z)_n}{(1 - z - n)_k}, \\
(z)_{mn} = m^m n \prod_{j=1}^m \frac{(z + j - 1)_n}{m} 
\]

and

\[
(2z)_{2n} = 2^{2n}(\frac{z}{2})_n(\frac{z + 1}{2})_n.
\]

In terms of the incomplete gamma functions \( \gamma(s, x) \) and \( \Gamma(s, x) \) defined by \([1.1]\) and \([1.2]\), the incomplete Pochhammer symbols \((\lambda, x)_n\) and \([\lambda, x]_n\) are defined respectively as follows

\[
(\lambda, x)_n = \frac{\gamma(\lambda + n, x)}{\Gamma(\lambda)} 
\]

(1.5)

and

\[
[\lambda, x]_n = \frac{\Gamma(\lambda + n, x)}{\Gamma(\lambda)}. 
\]

(1.6)

These incomplete Pochhammer symbols \((\lambda, x)_n\) and \([\lambda, x]_n\) satisfy the following decomposition relation

\[
(\lambda, x)_n + [\lambda, x]_n = (\lambda)_n. 
\]

(1.7)

If \( A_p \) denotes the array of \( p \) parameters \( a_1, a_2, \ldots, a_p \), then the Pochhammer symbols \((A_p)_n\) and incomplete Pochhammer symbols \((A_p, x)_n\) and \([A_p, x]_n\) respectively are defined by

\[
(A_p)_n = (a_1)_n(a_2)_n \cdots (a_p)_n, 
\]

(1.8)

\[
(A_p, x)_n = (a_1, x)_n(a_2)_n \cdots (a_p)_n 
\]

(1.9)

and

\[
[A_p, x]_n = [a_1, x]_n(a_2)_n \cdots (a_p)_n, 
\]

(1.10)

with the aid of \([1.9]\) and \([1.10]\), the following decomposition formula holds

\[
(A_p, x)_n + [A_p, x]_n = \{(a_1, x)_n + [a_1, x]_n\}(a_2)_n \cdots (a_p)_n \\
= (a_1)_n(a_2)_n \cdots (a_p)_n = (A_p)_n \\
= (A_p)_n 
\]
The incomplete Pochhammer symbols and their application to generalized hypergeometric functions

(see [10]).

In 1933, Wright [26] has extended the generalization of the hypergeometric function in the following form

\[ p \Psi_q(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n) \cdots \Gamma(\alpha_p + \beta_p n)}{\Gamma(\rho_1 + \mu_1 n) \cdots \Gamma(\rho_q + \mu_q n)} , \tag{1.11} \]

where \( \beta_r \) and \( \mu_s \) are real positive numbers such that

\[ 1 + \sum_{s=1}^{q} \mu_s - \sum_{r=1}^{p} \beta_r > 0. \]

When \( \beta_r \) and \( \mu_s \) are equal to 1, equation (1.11) is differ from generalized hypergeometric function \( p F_q(z) \) by a constant multiplier only. This generalized form of hypergeometric function has been investigated by Malovichko [11] and others. Dotsenko [6] considered one of the interesting cases which has the following form

\[ 2 R_1^{\omega, \mu}(z) = 2 R_1(a, b; \omega; \mu; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a + n) \Gamma(b + \frac{\mu}{n})}{\Gamma(c + \frac{\omega}{n})} z^n n! \tag{1.12} \]

and its integral representation is expressed in the form

\[ 2 R_1^{\omega, \mu}(z) = \frac{\mu \Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_{0}^{1} t^{b-1} (1 - t^\mu)^{c-b-1} (1 - z t^\omega)^{-a} dt \tag{1.13} \]

where \( \text{Re}(c) > \text{Re}(b) > 0 \).

In 2001, Virchenko et al. [20] have investigated by direct observation, the function \( 2 R_1^{\omega, \mu}(z) \) is not symmetric with respect to the parameters \( a \) and \( b \). In the same paper, they defined the said Wright type hypergeometric function \( 2 R_1^\tau(z) \) in the following form

\[ 2 R_1^\tau(z) = 2 R_1(a, b; c; \tau; z) = 2 R_1^\tau \left[ \begin{array}{c} a, b; \\ \tau; c, z \end{array} \right] \]

\[ = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \sum_{n=0}^{\infty} \frac{(a + n) \Gamma(b + \tau n)}{\Gamma(c + \tau n)} z^n n! \quad \tau > 0, \quad |z| < 1 \tag{1.14} \]

and its integral representation is defined as

\[ 2 R_1^\tau(z) = 2 R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\tau \Gamma(b) \Gamma(c - b)} \int_{0}^{1} t^{b-1} (1 - t^\tau)^{c-b-1} (1 - z t)^{-a} dt \tag{1.15} \]

or

\[ 2 R_1^\tau(z) = 2 R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\tau \Gamma(b) \Gamma(c - b)} \int_{0}^{1} t^{\frac{1}{\tau}-1} (1 - t^\tau)^{c-b-1} (1 - z t)^{-a} dt. \tag{1.16} \]

Also, the same author defined the generalized hypergeometric function and their properties see [20, 21] in the following form

\[ 1 \phi_1^\tau(a; c; z) = \phi^\tau(a; c; z) = 1 \phi_1^\tau \left[ \begin{array}{c} a; \\ \tau; c, z \end{array} \right] \]
and we begin by introducing the generalized incomplete Gauss hypergeometric functions by
diffraction and plasma wave theory, fluid flow, nuclear and molecular physics, statistics and engineering. In this section, non trivial step as the closed form solution to a considerable number of problems in applied mathematics, astrophysics, the functions which are introduced in this section. The decomposition of incomplete gamma functions appear to be a matter of fact, several known properties of the generalized Gauss hypergeometric functions are recovered from those of incomplete Gauss hypergeometric functions. An important aspect of these generalized incomplete hypergeometric functions.

The Incomplete Generalized Gauss Hypergeometric Functions

Recently many researchers ([4], [7], [15], [16], [17], [19], [22, 23, 24, 25]) studied Wright type hypergeometric function and obtained some of its properties, a generalized inverse gaussian distribution with τ-generalization is the natural generalization of classical hypergeometric functions. Similarly, let p be the numerator parameters and q be the denominator parameters, then we can defined the generalized hypergeometric function \( _pR_q^\tau \) in the form

\[
P_R^\tau(z) = \frac{\Gamma(c)}{\tau \Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} e^{zt} dt,
\]

(1.18)

or

\[
P_R^\tau(z) = \frac{\Gamma(c)}{\tau \Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} e^{zt} dt,
\]

(1.19)

Similarly, let \( p \) be the numerator parameters and \( q \) be the denominator parameters, then we can defined the generalized hypergeometric function \( _pR_q^\tau \) in the form

\[
P_R^\tau(z) = \frac{\Gamma(c)}{\tau \Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} e^{zt} dt,
\]

(1.18)

or

\[
P_R^\tau(z) = \frac{\Gamma(c)}{\tau \Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} e^{zt} dt,
\]

(1.19)

where the \( \tau \)-generalization is the natural generalization of classical hypergeometric functions.

2 The Incomplete Generalized Gauss Hypergeometric Functions

The introduction of the incomplete Pochhammer symbols \((\lambda,x)_n\) and \([\lambda,x]_n\) in section [4] is useful in defining the incomplete Gauss hypergeometric functions. An important aspect of these generalized incomplete hypergeometric functions is that they enjoy most of the properties inherited by the generalized Gauss hypergeometric functions. As a matter of fact, several known properties of the generalized Gauss hypergeometric functions are recovered from those of the functions which are introduced in this section. The decomposition of incomplete gamma functions appear to be a non trivial step as the closed form solution to a considerable number of problems in applied mathematics, astrophysics, diffraction and plasma wave theory, fluid flow, nuclear and molecular physics, statistics and engineering. In this section, we begin by introducing the generalized incomplete Gauss hypergeometric functions by

\[
\begin{align*}
2\gamma^\tau_1(a,x; b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{(a)_n \Gamma(b + n\tau)}{\Gamma(c + n\tau)} \frac{z^n}{n!}, \\
2\gamma^\tau_1(a,x; b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a,x]_n \Gamma(b + n\tau)}{\Gamma(c + n\tau)} \frac{z^n}{n!}.
\end{align*}
\]

(2.1)
With the aid of (2.1) and (2.2), the following decomposition formula holds for the generalized Gauss hypergeometric function

\[
2\Gamma_1^+(a, x; b; z; c; z) + 2\Gamma_1^-(a, x; b; z; c; z) = 2R_1^{+} \left( a, b; z; c; z \right). \tag{2.3}
\]

**Theorem 2.1** The following integral representation holds true

\[
2\Gamma_1^+(a, x; b; z; c; z) = \frac{1}{\Gamma(a)} \int_{x}^{\infty} t^{a-1} e^{-t} \times_1 \phi_1^+ \left( b; zt; c; z \right) dt, \quad x \geq 0, R(a) > 0. \tag{2.4}
\]

**Proof.** By definition, we have

\[
2\Gamma_1^+(a, x; b; z; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{[a, x]_n \Gamma(b + n\tau) z^n}{n!}.
\]

Using the integral representation of Pochhammer symbol \([a, x]_n\) in above equation, we get

\[
2\Gamma_1^+(a, x; b; z; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b + n\tau) z^n}{n!} \int_{x}^{\infty} t^{a-1-n} e^{-t} dt,
\]

which can be written as

\[
2\Gamma_1^+(a, x; b; z; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b + n\tau) (zt)^n}{n!} \int_{x}^{\infty} t^{a-1} e^{-t} dt
\]

\[
= \frac{1}{\Gamma(a)} \int_{x}^{\infty} t^{a-1} e^{-t} \times_1 \phi_1^+ \left( b; zt; c; z \right) dt.
\]

**Corollary 2.2** The following integral representation holds true for the generalized Gauss hypergeometric function

\[
2R_1^{+} \left( a, b; zt; c; z \right) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} t^{a-1} e^{-t} \times_1 \phi_1^+ \left( b; zt; c; z \right) dt. \tag{2.5}
\]

**Proof.** It follows from the assertion (2.4) of theorem 2.1 by setting \(x = 0\).

**Theorem 2.3** The following derivative formula holds true

\[
\frac{d}{dz} \left\{ 2\Gamma_1^+(a, x; b; z; c; z) \right\} = \frac{(a)_n \Gamma(c) \Gamma(b + n\tau)}{\Gamma(b) \Gamma(c + n\tau)} \times_2 \Gamma_1^+ \left( a + n, x; b + n\tau; c + n\tau; z \right). \tag{2.6}
\]

**Proof.** By differentiating both sides of (2.2) with respect to \(z\), we observe that

\[
\frac{d}{dz} \left\{ 2\Gamma_1^+(a, x; b; z; c; z) \right\} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{[a, x]_n \Gamma(b + n\tau)}{\Gamma(c + n\tau)} \frac{z^{n-1}}{(n-1)!}.
\]
which follows
\[
\frac{d}{dz} \left\{ T_1(a, x; b; c; z) \right\} = \frac{\Gamma(c) \sum_{n=0}^{\infty} [a, x]_{n+1} \Gamma(b + n \tau + \tau) z^n}{\Gamma(b) \Gamma(c + n \tau + \tau) n!}
\]
\[
= \frac{a \Gamma(c) \Gamma(b + \tau) \Gamma(c + \tau)}{\Gamma(b) \Gamma(c + c + n \tau + \tau)} \sum_{n=0}^{\infty} \frac{[a + 1, x]_n \Gamma(b + n \tau + \tau) z^n}{\Gamma(c + n \tau + \tau) n!}
\]
\[
= \frac{a \Gamma(c) \Gamma(b + \tau)}{\Gamma(b) \Gamma(c + \tau)} \times 2^{\tau} \Gamma_1 \left[ \begin{array}{c} a + 1, x; b + \tau; c + \tau; \\
\end{array} \right] \frac{dz}{d^2 b};
\]
which is (2.6) for \( n = 1 \). Thus the required result follows by Mathematical induction on \( n \).

**Corollary 2.4** The following derivative formula holds true for generalized Gauss hypergeometric function
\[
\frac{d^n}{dz^n} \left\{ T_1(a, x; b; c; z) \right\} = \frac{(a)_n \Gamma(c) \Gamma(b + n \tau + \tau)}{\Gamma(b) \Gamma(c + n \tau + \tau)} \times 2^{\tau} \Gamma_1 \left[ \begin{array}{c} a + n, b + n \tau; c + n \tau; \\
\end{array} \right]. \quad (2.7)
\]
**Proof.** By setting \( x = 0 \) in (2.6), we get the required result.

**Theorem 2.5** The following recurrence relation holds true
\[
[\tau - (c - 1)] \Gamma_1 \left[ \begin{array}{c} a, b; c; z \\
\end{array} \right] = b_2 \Gamma_1 \left[ \begin{array}{c} a + 1, x; b + 1; c; z \\
\end{array} \right] - (c - 1) \Gamma_1 \left[ \begin{array}{c} a, x; b; c - 1; z \\
\end{array} \right]. \quad (2.8)
\]
**Proof.** Using the following contiguous function relation for generalized confluent hypergeometric function (see [20] [21]
\[
(c - b - 1) \phi^z \left[ \begin{array}{c} b; c; z \\
\end{array} \right] = (c - 1) \phi^z \left[ \begin{array}{c} b; c - 1; z \\
\end{array} \right] - b \phi^z \left[ \begin{array}{c} b + 1; c; z \\
\end{array} \right],
\]
we find from (2.4) that
\[
[\tau - (c - 1)] \Gamma_1 \left[ \begin{array}{c} a, x; b; c; z \\
\end{array} \right] = \frac{1}{\Gamma(a)} \int_{x^z}^{\infty} e^{-t} \left( (c - 1) \phi^z \left[ \begin{array}{c} b; c - 1; z \\
\end{array} \right] - b \phi^z \left[ \begin{array}{c} b + 1; c; z \\
\end{array} \right] \right) dt
\]
which can be rewritten as follows
\[
[\tau - (c - 1)] \Gamma_1 \left[ \begin{array}{c} a, x; b; c; z \\
\end{array} \right] = (c - 1) \Gamma_1 \left[ \begin{array}{c} a + 1, x; b; c - 1; z \\
\end{array} \right] - b_2 \Gamma_1 \left[ \begin{array}{c} a, x; b + 1; c; z \\
\end{array} \right].
\]

**Corollary 2.6** (see [20]) The following recurrence relation holds true for generalized Gauss hypergeometric function
\[
[\tau - (c + 1)] \Gamma_1 \left[ \begin{array}{c} a, x; b; c; z \\
\end{array} \right] = b_2 \Gamma_1 \left[ \begin{array}{c} a + 1, x; b + 1; c; z \\
\end{array} \right] - (c - 1) \Gamma_1 \left[ \begin{array}{c} a, x; b; c - 1; z \\
\end{array} \right]. \quad (2.9)
\]
**Proof.** The well known recurrence relation follows from (2.8) by putting \( x = 0 \).

**Theorem 2.7** The following recurrence relation holds true
\[
\frac{a \tau z \Gamma(c) \Gamma(b + \tau)}{\Gamma(b + 1) \Gamma(c + \tau) 2^{\tau} \Gamma_1 \left[ \begin{array}{c} a + 1, x; b + \tau; c + \tau; z \\
\end{array} \right] = 2^{\tau} \Gamma_1 \left[ \begin{array}{c} a, x; b; c; z \\
\end{array} \right] - 2^{\tau} \Gamma_1 \left[ \begin{array}{c} a, x; b; c; z \\
\end{array} \right]. \quad (2.10)
\]
Proof. We note that
\[ a\tau z\Gamma(c)\Gamma(b + \tau)_{2}\Gamma_{1}^{r} \left[ \begin{array}{c} (a + 1, x), \ b + \tau; \\ c + \tau; \\ z \end{array} \right] = \frac{a}{\Gamma(a + 1)} \int_{0}^{x} t^{a-1}e^{-t} \left( \Gamma(z\tau\Gamma(c)\Gamma(b + \tau)\phi^{r} \left[ \begin{array}{c} b + \tau; \\ c + \tau; \\ z \end{array} \right] \right) dt. \tag{2.11} \]

Now, by using the following known contiguous relation [4]
\[ \Gamma(b)\Gamma(c + \tau)\phi^{r} \left[ \begin{array}{c} b; \\ c; \\ z \end{array} \right] + \Gamma(b)\Gamma(c + \tau)\phi^{r} \left[ \begin{array}{c} b - 1; \\ c; \\ z \end{array} \right] = \tau z\Gamma(c)\Gamma(b + \tau - 1)\phi^{r} \left[ \begin{array}{c} b + \tau - 1; \\ c + \tau; \\ z \end{array} \right], \tag{2.12} \]

hence (2.11) can be rewritten as
\[ a\tau z\Gamma(c)\Gamma(b + \tau)_{2}\Gamma_{1}^{r} \left[ \begin{array}{c} (a + 1, x), \ b + \tau; \\ c + \tau; \\ z \end{array} \right] = \frac{\Gamma(b + 1)\Gamma(c + \tau)}{\Gamma(a)} \times \int_{x}^{e} t^{a-1}e^{-t} \left( \phi^{r} \left[ \begin{array}{c} b + 1; \\ c; \\ zt \end{array} \right] - \phi^{r} \left[ \begin{array}{c} b; \\ c; \\ zt \end{array} \right] \right) dt. \tag{2.13} \]

From (2.4) and (2.11), we get
\[ a\tau z\Gamma(c)\Gamma(b + \tau)_{2}\Gamma_{1}^{r} \left[ \begin{array}{c} (a + 1, x), \ b + \tau; \\ c + \tau; \\ z \end{array} \right] = \Gamma(b + 1)
\times \Gamma(c + \tau) \left[ 2\Gamma_{1}^{r} \left[ \begin{array}{c} (a, x), \ b + 1; \\ c; \\ z \end{array} \right] - 2 \Gamma_{1}^{r} \left[ \begin{array}{c} (a, x), \ b; \\ c; \\ z \end{array} \right] \right]. \]

Corollary 2.8 (see [7]) The following well known recurrence relations holds true for generalized Gauss hypergeometric function
\[ a\tau z\Gamma(c)\Gamma(b + \tau)_{2}R_{1}^{r} \left[ \begin{array}{c} a + 1, \ b; \\ c + \tau; \\ z \end{array} \right] = \Gamma(b + 1)\Gamma(c + \tau) \left( 2R_{1}^{r} \left[ \begin{array}{c} a, \ b + 1; \\ c; \\ z \end{array} \right] - 2R_{1}^{r} \left[ \begin{array}{c} a, \ b; \\ c; \\ z \end{array} \right] \right). \tag{2.10} \]

Proof. The above corollary follows from (2.10) by putting x = 0.

Theorem 2.9 The following integral representation holds true
\[ 2\Gamma_{1}^{r} \left[ \begin{array}{c} (a, x), \ b; \\ c; \\ z \end{array} \right] = \frac{1}{\tau B(b, c - b)} \int_{0}^{1} t^{a-1}(1 - t^{b})^{c-b-1} x \Gamma_{r}^{r} \left[ \begin{array}{c} (a, x); \\ z; \\ - \end{array} \right] dt, R(c) > R(b) > 0. \tag{2.14} \]

Proof. We note that
\[ \frac{\Gamma(b + n\tau)\Gamma(c)}{\Gamma(b)\Gamma(c + n\tau)} = \frac{\Gamma(b + n\tau)\Gamma(c - b)}{\Gamma(b)\Gamma(c + n\tau)\Gamma(c - b)} = \frac{B(b + n\tau, c - b)}{B(b, c - b)}. \]

This implies
\[ \frac{\Gamma(b + n\tau)\Gamma(c)}{\Gamma(b)\Gamma(c + n\tau)} = \frac{1}{B(b, c - b)} \int_{0}^{1} t^{a-1}(1 - t)^{c-b-1}(t^{\tau})^{n} dt, R(c) > R(b) > 0. \]
We observe that
\[
2\Gamma_1^\tau \left[ \begin{array}{c} (a, x), \ b; \\ c; \ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[a, x]_n \Gamma(b + n \tau) z^n}{\Gamma(c + n \tau)} \frac{1}{n!} \\
= \frac{1}{B(b, c - b)} \sum_{n=0}^{\infty} [a, x]_n \frac{(zt^n)^n}{n!} \int_0^1 t^{b-1}(1-t)^{c-b-1}(t^n)^n dt \\
= \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times 1_0^{\tau} \left[ (a, x); -zt \right] dt
\]
or
\[
2\Gamma_1^\tau \left[ \begin{array}{c} (a, x), \ b; \\ c; \ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[a, x]_n \Gamma(b + n \tau) z^n}{\Gamma(c + n \tau)} \frac{1}{n!} \\
= \frac{1}{B(b, c - b)} \sum_{n=0}^{\infty} [a, x]_n \frac{(zt^n)^n}{n!} \int_0^1 t^{b-1}(1-t)^{c-b-1}(t^n)^n dt \\
= \frac{1}{\tau B(b, c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times 1_0^{\tau} \left[ (a, x); -zt \right] dt
\]

**Corollary 2.10** The following well known integral representation holds true for generalized Gauss hypergeometric function
\[
2R_1^\tau \left[ \begin{array}{c} a, \ b; \\ c; \ z \end{array} \right] = \frac{1}{\tau B(b, c - b)} \int_0^1 t^{b-1}(1-t^{\frac{1}{\tau}})^{c-b-1}(1-zt)^{-a} dt, R(c) > R(b) > 0. \quad (2.15)
\]

**Proof.** The above known integral representation follows immediately from [2.14] when we put \( x = 0 \) and use the relation
\[
1\Gamma_0^\tau \left[ \begin{array}{c} (a, 0); \\ -zt \end{array} \right] = (1-zt)^{-a}
\]

**Theorem 2.11** Let \( \tau \in \mathbb{R}^+ = (0, \infty), a, b, c \in \mathbb{C}, R(b) > 0, R(c) > 0. \) Then \( 2\Gamma_1^\tau \left[ \begin{array}{c} (a, x), \ b; \\ c; \ z \end{array} \right] \) is represented by the Mellin-Barnes integral in the form
\[
2\Gamma_1^\tau \left[ \begin{array}{c} (a, x), \ b; \\ c; \ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(a - s, x)\Gamma(b - \tau s)}{\Gamma(c - \tau s)} (-z)^{-s} ds \quad (2.16)
\]
where \( |\arg(z)| < \pi \) and the contour of integration beginning at \(-i\infty\), and ending at \(+i\infty\), and intended to separate at \( s = -n, n = 0, 1, 2, \cdots \) to the left and all the poles at \( s = k + a, k = 0, 1, 2, \cdots \) as well as \( s = \frac{k+b}{\tau}, n = 0, 1, \cdots \) to the right.
Proof. Here, we shall use the sum of residues at the poles $s = -n$, $n = 0, 1, 2, \cdots$ to obtain the integral form (2.16).

\[
2\Gamma_1^r \left[ \begin{array}{c} (a, x), \; b; \\ c; \\ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[a, x]_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!}
\]

this implies that

\[
2\Gamma_1^r \left[ \begin{array}{c} (a, x), \; b; \\ c; \\ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a + n, x)\Gamma(b + \tau n)}{\Gamma(1 + n)\Gamma(c + \tau n)} (-z)^n.
\]

(2.17)

Now, the Mellin-Barnes integral is given as

\[
2\Gamma_1^r \left[ \begin{array}{c} (a, x), \; b; \\ c; \\ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int \frac{\Gamma(s)\Gamma(a - s, x)\Gamma(b - \tau s)}{\Gamma(c - \tau s)} (-z)^{-s} ds
\]

\[
= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \tau^{n} \left[ \frac{\Gamma(s)\Gamma(a - s, x)\Gamma(b - \tau s)}{\Gamma(c - \tau s)} \right] (-z)^{-s}
\]

\[
= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int \frac{\Gamma(s)\Gamma(a - s, x)\Gamma(b - \tau s)}{\Gamma(c - \tau s)} (-z)^{-s} ds
\]

This implies that

\[
2\Gamma_1^r \left[ \begin{array}{c} (a, x), \; b; \\ c; \\ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a + n, x)\Gamma(b + \tau n)}{\Gamma(1 + n)\Gamma(c + \tau n)} (-z)^n.
\]

(2.18)

Thus from (2.17) and (2.18), we get the required result.

Corollary 2.12 (see [16]) Let $\tau \in \mathbb{R}^+ = (0, \infty)$, $a, b, c \in \mathbb{C}$, $R(b) > 0, R(c) > 0$. Then the well known Mellin-Barnes integral $2R_1^r \left[ \begin{array}{c} a, \; b; \\ c; \\ z \end{array} \right]$ is represented in the following form

\[
2\Gamma_1^r \left[ \begin{array}{c} (a, x), \; b; \\ c; \\ z \end{array} \right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int \frac{\Gamma(s)\Gamma(a - s)\Gamma(b - \tau s)}{\Gamma(c - \tau s)} (-z)^{-s} ds
\]

(2.19)

where $|\arg(z)| < \pi$.

Proof. The above known integral representation follows from (2.16) by putting $x = 0$. 

3 A Pair Of Generalized Incomplete Hypergeometric Functions

In this section, we define the generalized incomplete hypergeometric functions \( p^\gamma_q \) and \( p^\Gamma_q \), which are respectively defined as

\[
p^\gamma_q \left[ \begin{array}{c} (A_p, x); \\ B_q; \\ z \end{array} \right] = p^\gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] = \frac{\Gamma(b_1) \cdots \Gamma(b_q) \sum_{n=0}^{\infty} [a_1, x]_n \Gamma(a_2 + n\tau) \cdots \Gamma(a_p + n\tau) \ z^n}{\Gamma(a_2) \cdots \Gamma(a_p) \sum_{n=0}^{\infty} \Gamma(b_1 + n\tau) \Gamma(b_2 + n\tau) \cdots \Gamma(b_q + n\tau) \ n!}
\]

and

\[
p^\Gamma_q \left[ \begin{array}{c} (A_p, x); \\ B_q; \\ z \end{array} \right] = p^\Gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] = \frac{\Gamma(b_1) \cdots \Gamma(b_q) \sum_{n=0}^{\infty} \Gamma(b_1 + n\tau) \Gamma(b_2 + n\tau) \cdots \Gamma(b_q + n\tau) \ n!}{\Gamma(b_1) \cdots \Gamma(b_q) \sum_{n=0}^{\infty} \Gamma(b_1 + n\tau) \Gamma(b_2 + n\tau) \cdots \Gamma(b_q + n\tau) \ n!}
\]

where \( A_p \) and \( B_q \) are defined in section 1. The generalize incomplete hypergeometric functions \( p^\gamma_q \) and \( p^\Gamma_q \) satisfy the following decomposition formula

\[
p^\gamma_q \left[ \begin{array}{c} (A_p, x); \\ B_q; \\ z \end{array} \right] + p^\Gamma_q \left[ \begin{array}{c} (A_p, x); \\ B_q; \\ z \end{array} \right] = p^R_q \left[ \begin{array}{c} A_p; \\ B_q; \\ z \end{array} \right] \quad \text{(3.1)}
\]

**Theorem 3.1** Then the following integral representation holds true

\[
p^\Gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] = \frac{1}{\Gamma(a)} \int_0^1 t^{a_1-1} e^{-t} \times p-1 \ R^\gamma_q \left[ \begin{array}{c} a_2,\ldots,a_p; \\ b_1,b_2,\ldots,b_q; \\ zt \end{array} \right] \ dt, x \geq 0. \quad \text{(3.2)}
\]

**Proof.** By definition, we have

\[
p^\Gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] = \frac{\Gamma(b_1) \cdots \Gamma(b_q) \sum_{n=0}^{\infty} [a_1, x]_n \Gamma(a_2 + n\tau) \cdots \Gamma(a_p + n\tau) \ z^n}{\Gamma(a_2) \cdots \Gamma(a_p) \sum_{n=0}^{\infty} \Gamma(b_1 + n\tau) \Gamma(b_2 + n\tau) \cdots \Gamma(b_q + n\tau) \ n!}
\]

now, by using the integral representation of \([a_1, x]_n\) in above equation, we get the required result.

**Corollary 3.2** The following integral representation holds true for generalized hypergeometric function

\[
p^R_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] = \frac{1}{\Gamma(a)} \int_0^\infty t^{a_1-1} e^{-t} \times p-1 \ R_q^\gamma \left[ \begin{array}{c} a_2,\ldots,a_p; \\ b_1,b_2,\ldots,b_q; \\ zt \end{array} \right] \ dt, R(a_1) > 0. \quad \text{(3.3)}
\]

**Proof.** By putting \( x = 0 \) in (3.2), we get the required result. Similarly, we can prove the following differential and integral representations.

**Theorem 3.3** The following derivative formula holds true for generalized incomplete hypergeometric function

\[
\frac{d^n}{dx^n} \left\{ p^\Gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p; \\ b_1,b_2,\ldots,b_q; \\ z \end{array} \right] \right\} = \frac{(a_1)_n \Gamma(a_2 + n\tau) \cdots \Gamma(a_p + n\tau)}{\Gamma(b_1 + n\tau) \Gamma(b_2 + n\tau) \cdots \Gamma(b_q + n\tau)} \ R^\gamma_q \left[ \begin{array}{c} (a_1 + n, x), a_2 + n\tau, \ldots, a_p + n\tau; \\ b_1 + n\tau, b_2 + n\tau, \ldots, b_q + n\tau; \\ z \end{array} \right]. \quad \text{(3.4)}
\]
Corollary 3.4 The following derivative formula holds for generalized hypergeometric function

\[
\frac{d^n}{dz^n} \left\{ p R_q^\tau \left[ \begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ b_1, b_2, \cdots, b_q; \end{array} \right] \right\} = \frac{(a_1)_n \Gamma(a_2 + n \tau) \cdots \Gamma(a_p + n \tau)}{\Gamma(b_1 + n \tau) \Gamma(b_2 + n \tau) \cdots \Gamma(b_q + n \tau)} R_q^\tau \left[ \begin{array}{c} (a_1 + n, x), a_2 + n \tau, \cdots, a_p + n \tau; \\ b_1 + n \tau, b_2 + n \tau, \cdots, b_q + n \tau; \end{array} \right].
\]

(3.5)

Theorem 3.5 The following integral representation holds true

\[
P q^\tau \left[ \begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ b_1, b_2, \cdots, b_q; \end{array} \right] = \frac{1}{\tau B(a_p, b_q - a_p)} \int_0^1 t^{a_p - 1} (1 - t)^{b_q - a_p - 1} \left( e^{-\tau s} - 1 \right) B(0, t) dt,
\]

(3.6)

where \( R(b_q) > R(a_p) > 0, x \geq 0 \).

Corollary 3.6 The following integral representation holds true for \( k > 0 \)

\[
P q^\tau \left[ \begin{array}{c} a_1, a_2, \cdots, a_p; \\ b_1, b_2, \cdots, b_q; \end{array} \right] = \frac{1}{\tau B(a_p, b_q - a_p)} \int_0^1 t^{a_p - 1} (1 - t)^{k - a_p - 1} \left( e^{-\tau s} - 1 \right) B(0, t) dt,
\]

(3.7)

where \( R(b_q) > R(a_p) > 0, x \geq 0 \).

Theorem 3.7 Let \( \tau \in \mathbb{R}^+ = (0, \infty) \), \( a, b, c \in \mathbb{C} \), \( R(b) > 0, R(c) > 0 \). Then \( p \Gamma_q^\tau \left[ \begin{array}{c} (A_p, x), \\ B_q; \end{array} \right] \) is represented by the Mellin-Barnes integral in the form

\[
2 \Gamma_q \left[ \begin{array}{c} (a, x), b; \\ c; \end{array} \right] = \frac{\Gamma(b_1) \Gamma(b_2) \cdots \Gamma(b_q)}{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_p)} \frac{1}{2\pi i} \int L \frac{\Gamma(s) \Gamma(a_1 - s, x) \Gamma(a_2 - s) \cdots \Gamma(a_p - s)}{\Gamma(b_1 - s) \Gamma(b_2 - s) \cdots \Gamma(b_q - s)} (-z)^{-s} ds
\]

(3.8)

where \( |\arg(z)| < \pi \) and the contour of integration beginning at \( -i\infty \), and ending at \( +i\infty \), and intended to separate at \( s = -n, n = 0, 1, 2, \cdots \) to the left and all the poles at \( s = k + a, k = 0, 1, 2, \cdots \) as well as \( s = \frac{k+b}{2}, n = 0, 1, \cdots \) to the right.

Corollary 3.8 Let \( \tau \in \mathbb{R}^+ = (0, \infty) \), \( a, b, c \in \mathbb{C} \), \( R(b) > 0, R(c) > 0 \). Then the well known Mellin-Barnes integral \( p R_q^\tau \)

\[
p R_q^\tau \left[ \begin{array}{c} (a, x), b; \\ c; \end{array} \right] = \frac{\Gamma(b_1) \Gamma(b_2) \cdots \Gamma(b_q)}{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_p)} \frac{1}{2\pi i} \int L \frac{\Gamma(s) \Gamma(a_1 - s, x) \Gamma(a_2 - s) \cdots \Gamma(a_p - s)}{\Gamma(b_1 - s) \Gamma(b_2 - s) \cdots \Gamma(b_q - s)} (-z)^{-s} ds
\]

(3.9)

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