On the Time Fractional Modulation for Electron Acoustic Shock Waves *

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Nonlinear features of electron-acoustic shock waves are studied. The Burgers equation is derived and converted to the time fractional Burgers equation by Agrawal’s method. Using the Adomian decomposition method, the shock wave solutions of the time fractional Burgers equation are constructed. The effect of time fractional parameter on the shock wave properties in auroral plasma is investigated.

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Recently, applications of fractional nonlinear partial differential equations have received major attention in fluid mechanics and physics plasmas.[1–4] Electron acoustic wave (EAW) propagation in plasmas has been studied experimentally and theoretically.[5–8] Clearly, evolution of small amplitude solitary nonlinear structures in fluids and plasmas has been investigated by using many nonlinear equations (NLEs).[9–15] These equations are obtained by using perturbation methods such as the well-known renormalization group. Using this method, the shock wave solutions of the time fractional Burgers equation (TFB) equation are constructed. The effect of time fractional parameter on the shock wave properties in multi-ion plasmas is studied. It is noted that the acoustic shocks are modified by the parameter of heavy-to-light ion number density. Also, both polarity potentials exist in the plasma system.[29] The dust-acoustic shock properties have been studied in two-temperature dust plasmas using the Burgers equation with time fractional order.[30] Researchers discussed the time fractional parameter effect on shock wave features using the variational iteration technique. El-Shewy et al. studied shock waves using the space-time fractional KdV Burgers equation. It is noted that the parameter of space time fractional affects the coexistence of shocks.[15] Recently, shock wave modulation in viscous ion plasmas have been examined via higher order dissipation in Earth’s ionosphere.[31] In this study, an electron acoustic model with superthermal hot electrons is considered. The KdV equation is derived, Agrawal’s process[4,32] is used to derive the time fractional TFKdV equation, and the Adomian decomposition method[33,34] is used to solve them.

In this model, a three-component collisionless unmagnetized plasma having ions in the stationary state, viscous fluid of cold electron and κ velocity distributed hot electrons is studied. The normalized equations in a small, finite amplitude are given by

\[
\begin{align*}
\frac{\partial n_c(x,t)}{\partial t} + \frac{\partial n_c(x,t)u_c(x,t)}{\partial x} &= 0, \\
\frac{\partial u_c(x,t)}{\partial t} + u_c(x,t)\frac{\partial u_c(x,t)}{\partial x} &= \frac{\partial \phi(x,t)}{\partial x} + \frac{\eta}{\kappa} \frac{\partial^2 u_c(x,t)}{\partial x^2}, \\
\beta n_c + \frac{(2\kappa - 1)}{(2\kappa - 3)} \phi(x,t) &+ \frac{(2\kappa - 1)(2\kappa + 1)}{(2\kappa - 3)^2} \phi^2(x,t) - \beta = 0,
\end{align*}
\]

where \( n_c(x,t) \) is the density of cold fluid (superthermal hot) electron normalized by \( n_0(n_{th}) \), \( \phi \) is the electric potential (normalized by \( k_B/T_{th} \)), \( u_c(x,t) \) is the velocity of cold electron fluid (normalized by)
which are valid for conditions \( u = \) constant. The normalized kinematic viscosity reads

\[ \eta = \frac{\rho_{\text{num}}}{\rho_{\text{ball}}}, \]

where \( \rho_{\text{num}} \) is the electron density, \( \rho_{\text{ball}} \) is the electron density in terms of the thermal viscosity \( \mu \).

We have also defined the ratio parameter

\[ \beta = \frac{\mu_{\text{num}}}{\mu_{\text{ball}}}, \]

where the superthermal information[35,36] is now ‘hidden’ in the coefficients.

The stretched co-ordinates are introduced by the (RPT) method,[37]

\[ \tau = \frac{\epsilon^2 t}{\lambda}, \quad \xi = \frac{\epsilon^{1/2} (x - \lambda t)}{\lambda}, \]

where \( \epsilon \) is the speed of wave \( \lambda \) and is a small parameter. By expanding quantities in Eq. (1) about their equilibrium values, we obtain

\[
\begin{align*}
n_c(\xi, \tau) &= 1 + e u_1(\xi, \tau) + e^2 n_2(\xi, \tau) + \ldots, \\
u_c(\xi, \tau) &= e u_1(\xi, \tau) + e^2 u_2(\xi, \tau) + \ldots, \\
\phi(\xi, \tau) &= e \phi_0(\xi, \tau) + e^2 \phi_2(\xi, \tau) + \ldots,
\end{align*}
\]

which are valid for conditions \( |\xi| \to \infty, n = n_c = 1, u = 0 \) and \( \phi = 0 \). Substituting Eqs. (4) and (5) into Eq. (1), the lowest order in \( \epsilon \) is given by

\[
n_1(\xi, \tau) = -\frac{g_1 \phi_0(\xi, \tau)}{\beta}, \quad u_1(\xi, \tau) = -\frac{\phi_1(\xi, \tau)}{\lambda}.
\]

The dispersion form is given by

\[
g_1 \frac{\lambda^2 - \beta}{\beta \lambda} = 0,
\]

where

\[
g_1 = \frac{2\kappa - 1}{2\kappa - 3}, \quad g_2 = \frac{g_1(2\kappa + 1)}{(2\kappa - 3)}.
\]

The next equations of \( O(\epsilon^2) \) yield

\[
\begin{align*}
-\lambda \frac{\partial n_2(\xi, \tau)}{\partial \xi} + &\frac{\partial u_2(\xi, \tau)}{\partial \xi} + \frac{\partial n_1(\xi, \tau)}{\partial \tau} + \frac{\partial u_1(\xi, \tau)}{\partial \tau} = 0, \\
+ n_1(\xi, \tau) u_1(\xi, \tau) + &\frac{\partial u_1(\xi, \tau)}{\partial \xi} = 0, \\
-\lambda \frac{\partial u_2(\xi, \tau)}{\partial \xi} + &\frac{\partial u_1(\xi, \tau)}{\partial \tau} + \frac{\partial \phi_2(\xi, \tau)}{\partial \xi} + u_1(\xi, \tau) \frac{\partial u_1(\xi, \tau)}{\partial \tau} - \eta \frac{\partial^2 u_1(\xi, \tau)}{\partial \xi^2} = 0, \\
g_2 \phi_2^2(\xi, \tau) + &g_1 \phi_2(\xi, \tau) + \beta n_2(\xi, \tau) = 0.
\end{align*}
\]

Putting \( n_2, \phi_2 \) and \( u_2 \) into Eqs. (9)–(11), one can obtain the Burger equation for \( \phi_1 \) as follows:

\[
\frac{\partial \phi_1(\xi, \tau)}{\partial \tau} + A \phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi, \tau)}{\partial \xi^2} = 0,
\]

where

\[
A = -\beta + 2g_2 \lambda^4 + 2g_1 \lambda^2, \quad B = -\frac{\beta \eta}{\beta + g_1 \lambda^2}.
\]

Equation (12) admits the EA shock wave solution as

\[
\phi_1(\xi, \tau) = \frac{2B}{A} [1 + \tanh(\xi - 2B\tau)],
\]

whose amplitude equals \( \frac{2B}{A} \). In Eq. (12), \( \phi_1(\xi, \tau) \) is a field variable, \( \tau \in T(= [0, T_f]) \) is the time variable, and \( \xi \) is a space coordinate. El-Wakil et al.[38] derived the time fractional Burgers equation in time, and in one space dimension we have the form of

\[
\begin{align*}
\frac{\partial}{\partial \tau} \phi_1(\xi, \tau) + A \phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi, \tau)}{\partial \xi^2} = 0,
\end{align*}
\]

where the Riesz fractional order is expressed as

\[
\begin{align*}
\frac{\partial}{\partial \tau} \phi_1(\xi, \tau) = \frac{1}{2} \int_0^\tau \frac{1}{1 - \alpha} \left[ \frac{d}{d \tau} \int_0^\tau dt \right] \phi_1(\xi, \tau),
\end{align*}
\]

which is valid for conditions \( |\tau| \to \infty, n = n_c = 1, u = 0 \) and \( \phi = 0 \). Substituting Eqs. (12) and (13) into Eq. (1), the lowest order in \( \epsilon \) is given by

\[
\begin{align*}
\frac{\partial \phi_1(\xi, \tau)}{\partial \tau} + &A \phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi, \tau)}{\partial \xi^2} = 0,
\end{align*}
\]

and the nonlinear term \( \theta(\psi(\xi, \tau)) \) is represented by the Adomian series as

\[
\begin{align*}
\theta(\psi(\xi, \tau)) = \sum_{n=0}^\infty A_n,
\end{align*}
\]

where \( A_n \) are the Adomian polynomials given by

\[
\begin{align*}
A_0 = \theta(\psi_0(\xi, \tau)), \\
A_1 = \psi_0 \frac{\partial \theta(\psi_0)}{\partial \psi_0}, \\
A_2 = \psi_0 \frac{\partial \theta(\psi_0)}{\partial \psi_0} + \frac{1}{2} \psi_0^2 \frac{\partial^2 \theta(\psi_0)}{\partial \psi_0^2},
\end{align*}
\]

and other polynomials can be generated in the same manner. Applying the operator \( \frac{\partial}{\partial \tau} \phi_1(\xi, \tau) \) on both sides of Eq. (15) yields

\[
\psi(\xi, \tau) = \sum_{n=0}^\infty \psi_n(\xi, \tau),
\]

(17a)

and the nonlinear term \( \theta(\psi(\xi, \tau)) \) is represented by the Adomian series as

\[
\theta(\psi(\xi, \tau)) = \sum_{n=0}^\infty A_n,
\]

(17b)

where \( A_n \) are the Adomian polynomials given by

\[
\begin{align*}
A_0 = \theta(\psi_0(\xi, \tau)), \\
A_1 = \psi_0 \frac{\partial \theta(\psi_0)}{\partial \psi_0}, \\
A_2 = \psi_0 \frac{\partial \theta(\psi_0)}{\partial \psi_0} + \frac{1}{2} \psi_0^2 \frac{\partial^2 \theta(\psi_0)}{\partial \psi_0^2},
\end{align*}
\]

(18a)

(18b)

(18c)
where $R D^{-\alpha}_\tau$ is the Riemann–Liouville fractional integral, which is defined as \cite{38,39}

$$ R D^{-\alpha}_\tau = \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{g(\tau')}{(\tau - \tau')^{1-\alpha}} d\tau', \quad 0 < \alpha < 1. \quad (20) $$

Substituting Eq. (17) into Eq. (19) leads to

$$ \sum_{n=0}^\infty \phi_{1n}(\xi, \tau) = \phi_1(\xi, 0) - R D^{-\alpha}_\tau \left. \left[ A\phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi, \tau)}{\partial \xi^2} \right] \right|_{\tau=0}. \quad (21) $$

The components $\phi_{1n}(\xi, \tau)$ of the solution $\phi_1(\xi, 0)$ can be computed using the following recursive relation

$$ \phi_{10}(\xi, 0) = \phi_1(\xi, 0), \quad (22a) $$

$$ \phi_{1k+1}(\xi, \tau) = - R D^{-\alpha}_\tau \left[ A\phi_1(\xi, \tau) \frac{\partial \phi_1(\xi, \tau)}{\partial \xi} + B \frac{\partial^2 \phi_1(\xi, \tau)}{\partial \xi^2} \right], \quad k \geq 1. \quad (22b) $$

The initial condition will be taken as

$$ \phi_1(\xi, 0) = \phi_m[1 + \tanh(c\xi)], \quad (23a) $$

where the amplitude ($\phi_m$) and the width ($c$) of the shock wave whose values depend on the physical parameters of the system are given by

$$ \phi_m = \nu/A, \quad c = \nu/2B. \quad (23b) $$

Substituting with this zero order in the recursive relation (22b) leads to the first recursive

$$ \phi_{11}(\xi, \tau) = -cA\phi_m^2\text{sech}(\xi)^2 \frac{\tau^\alpha}{\Gamma(\alpha + 1)}. \quad (24b) $$

Substituting this first order in the recursive relation (22b) gives the second order correction

$$ \phi_{12}(\xi, \tau) = -8B^2c^4\phi_m\text{sech}(\xi)^3 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)}. \quad (24c) $$

In the same manner, the higher recursive order can be calculated by using the Maple or the Mathematica package. Substituting from these different recursive orders into Eq. (17a) the solution of the time fractional Burgers equation is obtained as

$$ \phi_1(\xi, \tau) = \phi_1(\xi, 0) - cA\phi_m^2\text{sech}(\xi)^2 \frac{\tau^\alpha}{\Gamma(\alpha + 1)} $$

$$ -8B^2c^4\phi_m\text{sech}(\xi)^3 \frac{\tau^{2\alpha}}{\Gamma(2\alpha + 1)} + \ldots \quad (25) $$

The small amplitude electron-acoustic shock waves have been described by the time fractional Burgers Eq. (15). The propagation of electrostatic shocks in our model is caused by the wave dissipation due to the kinematic viscosity. Our motivation was to study effects of the fractional parameter $\alpha$, the kinematic viscosity coefficient $\eta$, the ratio of initial equilibrium density of cold electron to hot electrons $\beta$ and superthermal spectral index $\kappa$ of electrons on the propagating shocks. Numerical investigation values have been taken for auroral zone $E_0 \approx 100$ mV/m, $T_e \approx 5$ eV, $T_h \approx 250$ eV, $n_{e0} \approx 0.5$ cm$^{-3}$, $n_{ho} \approx 2.5$ cm$^{-3}$ (these parameters correspond to $\lambda_{De} \approx 7430$ cm). The comparison between shock solutions of the time fractional Burgers (TFBurgers) equation with the Adomian decomposition (ADM) method and the classical Burgers equation is shown in Fig. 1. The effect of the time fractional parameter $\alpha$ on the shock wave properties has been examined in Fig. 2. It is noted that introducing the time fractional increases the steepness of the shock wave. On the other hand, the effect of the ratio of initial equilibrium density of cold electron to hot electrons $\beta$ and superthermal parameter $\kappa$ on the amplitude and steepness of shock waves are depicted in Figs. 3 and 4. It is found that the increase of the parameters $\beta$ and $\kappa$ increases the steepness and the amplitude of the formed shock waves. Finally, the increase of the kinematic viscosity coefficient $\eta$ decreases the steepness and amplitude of the electron acoustic shock waves as shown in Fig. 5.
In summary, the time fractional factor $\alpha$ would modulate the shape and existence of shock wave profile. Also, it has been shown that the obtained shock wave is sensitive to the superthermal parameter $\kappa$, the ratio of initial equilibrium density of cold electron to hot electrons $\beta$ and the kinematic viscosity coefficient $\eta$. Furthermore, parameter $\alpha$ plays an effective role in theoretical modulation in shock structures to agree with the observation data in the auroral region.[5,8–11]

References