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ABSTRACT Different from a Newtonian fluid, couple stress fluid (CSF) includes a new material constant, which is responsible for couple stress and the lubricant viscosity. This material constant comes with the fourth-order spatial derivative term, and due to this higher-order derivative term in the momentum equation, this fluid (CSF) is comparatively less investigated even for the classical fluid problems. This paper aims to study the fractional model of CSF, based on the Atangana–Baleanu (AB) fractional derivatives definition. Since this AB definition is new, therefore, for the sake of comparison and correctness, this problem is also solved using the Caputo–Fabrizio (CF) fractional derivative definition. The CSF is considered to flow between two parallel plates, one of which is at rest and the other is moving with constant velocity. The external pressure gradient is also applied. This type of flow situations is usually called generalized Couette flow. The problem is first written in dimensionless form and then solved for the exact solution using the Laplace transform and the finite Fourier sine transform. The CSF velocity obtained via an AB fractional derivative is compared with the CSF velocity obtained via a CF fractional derivative approach, and the results obtained are shown graphically. The CSF results for interesting fluid parameters are displayed in various graphs for both the AB and CF fractional derivatives. It is observed that the CSF velocities obtained with the AB and CF fractional derivatives are the same for unit time. For time less than one and greater than one, variation in CSF velocities is observed. In limiting sense, the present CSF solutions are reduced to a similar Newtonian fluid problem solution in the absence of external pressure gradient.

INDEX TERMS Couple stress fluid, AB and CF, generalized Couette flow, Laplace transform, finite Fourier sine transform.

I. INTRODUCTION
Couple-stress fluids (CSF) contain a new material constant which makes them different from usual viscous fluids. The rheological properties of these fluids (CSFs) have many applications like, in the extraction process of crude oils from petroleum products, electrostatics precipitations, aerodynamics heating phenomenon, solidification process of liquid crystals, cooling of metallic plate in a bath, colloidal and suspension solutions [1]. Some interesting studies regarding non-Newtonian fluids can be found in [2], [3]. The concept of Couple stress fluid theory was developed by Stokes [4], where he considered Couple stresses in addition to the classical Cauchy stress. It is the simplest generalization of the classical theory of fluids which allows for polar effects such as the presence of couple stresses and body couples. CSF theory is explained in details by Stokes in his treatise theories of Fluids with Microstructure [5] wherein he also mentioned a list of different problems which were discussed by researchers related to Couple stress theory.
CFS models have remarkable applications in our daily life, like phenomena of pumping the fluids synthesis lubricants and biological process of the fluid, solidification of liquid crystals and animal blood. The CFS model has been considered by researchers for different scientific reasons and physical problems. Ramaniah [6] analyzed the CCFs and applied problem of lubrication like the squeeze film bearings between finite plates. Bujurke and Jayaraman [7] investigated the influence of Couple stresses in squeeze films. Furthermore, Lin [8], [9] also discussed the effects of Couple stresses in the cyclic squeeze films and characteristics between a sphere and a flat plate.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>CSF</td>
<td>Couple Stress fluid</td>
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<td>(d)</td>
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<td>(H(\tau))</td>
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<td>(CFD^\alpha_{\xi}(\cdot))</td>
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<td>(C_f(y, t))</td>
<td>Skin friction</td>
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<td>(C_f(1, t))</td>
<td>Skin friction for the upper plate</td>
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Devakar et al. [10] studied some fundamental steady flows of couple stress fluid, namely, Couette, Poiseuille and generalized Couette flow between parallel plates with slip conditions on the boundary. Furthermore, Devakar and Iyengar [11] examined run up the flow. In this type of flow, the motion of couple stress fluids between two moving parallel plates is initially induced by a constant pressure gradient. After a steady state is achieved, the pressure gradient is suddenly removed while the plates are impulsively started simultaneously. Sreenadh et al. [12] discussed MHD free convection flow of CSF in a vertical porous layer by employing the numerical technique. Farooq et al. [13] calculated steady Poiseuille flow of CSF through the perturbation technique. Hayat et al. [14] investigated the nonlinear unsteady CSF three-dimensional flow over a stretched surface considered the influence of mass transfer and chemical reaction in their study. Furthermore, there are many exact solutions for incompressible CSF have been discussed by Naeem [15], using a canonical transformation method. Ahmed et al. [16] calculated the oscillatory flows of a CSF in an inclined, rotating channel using complex variables. Many interesting problems regarding the CSFs or micropolar fluids can be found in the references [17]–[22].

The fluid flow between two parallel plates is called channel flow. An open channel is a waterway, canal or conduit in which a liquid flows with a free surface. Open channel flows are found in nature as well as in man-made structures. Open channel flow has many physical and practical applications like water-supply channels for irrigation, power supply, and drinking waters, conveyor channel in water treatment plants, storm waterways, some public fountains, culverts below roads and railways lines [23]. The flow between two parallel plates in which one plate is at rest and the other one is moving with constant velocity is called Couette flow. Furthermore, when an external pressure gradient is applied to the Couette flow the resultant flow is called generalized Couette flow. Generalized Couette flow has widely used in engineering and physical problems. Liu et al. [24] discussed the Couette flow of a generalized Oldroyd-B fluid with fractional derivative. McKeague and Khonsari [25] explained Generalized boundary interactions for powder lubricated Couette flows. Moser et al. [26] investigated Navier-Stokes equations with applications to Taylor-Couette flow. Makinde and Chinyoka [27] calculated numerically unsteady hydro-magnetic Generalized Couette flow of a reactive third-grade fluid with asymmetric convective cooling. The concept of fractional calculus is originated in 1695 when for the first time Leibniz introduced the notation for the nth order derivatives of a function, i.e. Leibniz writes a letter to Del Hospital in which he asked a question that what will happen if we take the order in fraction. After that researcher are attracted to developed different definitions of fractional derivatives. The researchers are motivated by the enormous very interesting novel applications of fractional calculus especially in physics, chemistry, biological science, engineering science, finance and other sciences which have been developed in recent years. The initial attempts of researchers in fractional calculus are collected in the book of Oldham and Spanier [28].

Fractional calculus has a wide range of industrial and engineering applications, for example, diffusion phenomena signal processing devices, diffusion problems, advection and dispersion of solutes in natural porous or fractured media [29]–[31]. Furthermore, Cuesta and Final [32] investigated image processing by means of a linear integrodifferential equation to visualize the image and image processing. Some other applications regarding fractional calculus are viscoelastic and viscoplastic materials under the external influences, bio-engineering, damping phenomena, and heat propagation are mentioned in the
references [33]–[36]. Due to these applications researchers are attracted to investigate different phenomena using fractional calculus and fractional order derivatives.

Recently, Michele Caputo and Mauro Fabrizio proposed a fractional operator on the basis of an exponential function, to overcome the issue of singularity problem of the kernel [37]. Caputo-Fabrizio fractional derivatives approach have no singular kernel. Many researchers are focused to investigate various physical problems by using CF fractional derivatives operator. Tahir et al. [38], investigated the unsteady flow of fractional Oldroyd-B fluids through rotating annulus. Sadiq et al. [39] studied some rotational flow of second grade fluid with Caputo-Fabrizio fractional derivative in an Annulus Al-Salti et al. [40], investigated the boundary-value problem for fractional heat equation involving Caputo-Fabrizio derivative. However, there were some difficulties. CF fractional approach has no singular kernel and there is non-locality problem in CF derivatives. To overcome these issues of the non-singular and non-locality of the kernel, Atangana and Nieto [41] and Atangana [42] proposed two new fractional derivative approaches based on the generalized Mittag-Leffler function. The concept of AB and CF fractional derivatives is applied by Nadeem et al. [43], to investigate a Casson fluid model with the combined effect of heat generation and chemical reaction. Akhtar and Shah [44], discussed exact solutions for some unsteady flows of a couple of stress fluid between parallel plates. Akhtar [45] considered CSF between two parallel plates using, Caputo time-fractional derivative and Caputo-Fabrizio time-fractional derivative.

The purpose of this article is to investigate unsteady CSF between the two parallel plates. In order to transform the classical model into time, the fractional model applies AB and CF time-fractional derivatives. The Laplace transform and classical model into time, the fractional model applies AB time-fractional derivative and Caputo-Fabrizio derivative. However, there were some difficulties. CF fractional approach has no singular kernel and there is non-locality problem in CF derivatives. To overcome these issues of the non-singular and non-locality of the kernel, Atangana and Nieto [41] and Atangana [42] proposed two new fractional derivative approaches based on the generalized Mittag-Leffler function. The concept of AB and CF fractional derivatives is applied by Nadeem et al. [43], to investigate a Casson fluid model with the combined effect of heat generation and chemical reaction. Akhtar and Shah [44], discussed exact solutions for some unsteady flows of a couple of stress fluid between parallel plates. Akhtar [45] considered CSF between two parallel plates using, Caputo time-fractional derivative and Caputo-Fabrizio time-fractional derivative.

The purpose of this article is to investigate unsteady CSF between the two parallel plates. In order to transform the classical model into time, the fractional model applies AB and CF time-fractional derivatives. The Laplace transform and Fourier sine transform technique is used to obtain the exact solutions for the present problem. Furthermore, the effect of various parameters is investigated on the fluid flow using different graphs.

II. FORMULATION AND SOLUTION

The laminar flow of an incompressible Couple stress fluid through a channel bounded by two horizontal infinite parallel plates is considered in this work. The fluid is taken along x-direction in the absence of body couples the continuity and momentum equations of the fluid are given as [44].

\[ \nabla \cdot \vec{V} = 0, \]

\[ \rho \frac{\partial \vec{V}}{\partial t} = - \nabla p - \mu \nabla \times \nabla \times \vec{V} - \eta_1 \nabla \times \nabla \times \nabla \times \vec{V} + \rho \vec{b}_1. \]  

where \( \rho \) represents the density of the fluid, \( \vec{V} \) represents velocity in vector form, \( \vec{b}_1 \) represents the body force vector, \( p \) represents the pressure, \( \mu \) is the dynamic viscosity and \( \eta_1 \) represents couple stress parameter.

The flow of unsteady CSF is considered unidirectional between the two parallel plates. Then the velocity field of the present flow is \( \vec{V} = (u(y, t), 0, 0) \) satisfies the equation of continuity (1) and the governing equation of the couple stress fluid in the absence of body forces \( \vec{b}_1 \), can be expressed in the following form:

\[ \rho \frac{\partial u(y, t)}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u(y, t)}{\partial y^2} - \eta_1 \frac{\partial^4 u(y, t)}{\partial y^4}. \]  

A. GENERALIZED COUETTE FLOW

The flow of incompressible laminar CSF is taken between two parallel plates separated by a distance \( d \). Furthermore, the upper plate is assumed to be at rest and the lower plate is moving with a uniform velocity \( U_0 \). Additionally, at the same time, a constant pressure gradient \( G \) is applied along the x-direction to the fluid as shown in Fig. 1.

With these assumptions the governing equation is given as [45]:

\[ \rho \frac{\partial u^s(y, t)}{\partial t} = G^s + \mu \frac{\partial^2 u^s(y, t)}{\partial y^2} - \eta_1 \frac{\partial^4 u^s(y, t)}{\partial y^4}. \]  

The following initial and boundary conditions are to be satisfied:

\[ \begin{align*}
  u^s(y, t) &= 0, & \text{for } 0 \leq y \leq d & \text{and } t = 0, \\
  u^s(y, t) &= U_0 H(t), & \text{for } y > 0 & \text{and } t > 0, \\
  \frac{\partial u^s(y, t)}{\partial y} &= 0, & \text{for } y = 0 & \text{and } y = d^s.
\end{align*} \]  

where \( H(t) \), represents Heaviside unit step function.

Introducing the following non-dimensional quantities

\[ \xi = \frac{y}{d} ; \quad u = \frac{u^s}{U_0} ; \quad \tau = \frac{U_0 t}{d} ; \quad Re = \frac{U_0 d}{v} ; \]

\[ \eta = \frac{\eta_1}{\mu d^2} \quad \text{and} \quad G = \frac{G^s}{\mu U^*} ; \]

after dimensionnalization we obtained the following initial and boundary value problem:

\[ \begin{align*}
  Re \frac{\partial u^s(\xi, \tau)}{\partial \tau} &= G + \frac{\partial^2 u^s(\xi, \tau)}{\partial \xi^2} - \eta \frac{\partial^4 u^s(\xi, \tau)}{\partial \xi^4}, \\
  u^s(\xi, \tau) &= 0, & \text{for } 0 \leq \xi \leq d & \text{and } \tau = 0, \\
  u^s(\xi, \tau) &= 1, & \text{for } \xi = 0 & \text{and } \tau > 0, \\
  u^s(\xi, \tau) &= 0, & \text{for } \xi = d & \text{and } \tau > 0, \\
  \frac{\partial u^s(\xi, \tau)}{\partial \xi} &= 0, & \text{at } \xi = 0 & \text{and } \xi = d.
\end{align*} \]
where $Re$ represents Reynolds number, $\eta$ represents the couple stress parameter and $G$ represents the external pressure gradient.

### III. Exact Solutions with Atangana-Baleanu Derivatives

To develop the AB fractional model for generalized Couette flow of CSF we replace partial derivatives with respect to $\tau$ and $\beta$ is the fractional operator of AB derivatives. Eq. (6) takes the following form:

$$AB D_\beta^\alpha Re u(\xi, \tau) = G + \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} - \eta \frac{\partial^4 u(\xi, \tau)}{\partial \xi^4},$$  

where $AB D_\beta^\alpha(\cdot)$ is the AB time-fractional derivatives of order $\beta$ which is defined as [42].

$$AB D_\beta^\alpha(\cdot) = \frac{1}{1-\beta} \int_0^\tau (\frac{\beta}{1-\beta}) f'(\tau) d\tau,$$

where $E_\beta$ is the generalized form of Mittag-Leffler function which is defined by [46].

$$E_\beta (-t^\beta) = \sum_{k=0}^{\infty} \frac{(-t)^{\beta k}}{\Gamma(\beta k + 1)}.$$

Applying the Laplace transform to Eq. (8) and incorporate the given initial condition from Eq. (7), we get the following form:

$$q^\beta Re u(\xi, \gamma) \left( q^\beta + H_1 \right) = \frac{G}{q} + \frac{d^2 u(\xi, \gamma)}{d \xi^2} - \eta \frac{d^4 u(\xi, \gamma)}{d \xi^4}.$$

The transformed boundary conditions from Eq. (7) can be written in the following form:

$$\bar{u}(\xi, \gamma) = \frac{1}{2}, \quad \text{for} \quad \xi = 0 \quad \text{and} \quad q > 0,$$

$$\pi(\xi, q) = 0, \quad \text{for} \quad \xi = d \quad \text{and} \quad q > 0,$$

$$\frac{\partial^2 u(\xi, q)}{\partial \xi^2} = 0, \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = d.$$

To apply sine Fourier transform we multiplying both sides of the Eq. (11) by $\sin\left(\frac{n \pi \xi}{d}\right)$ and taking the limits of integration from 0 to $d$ with respect to $\xi$ and incorporate Eq. (12), we obtain:

$$\bar{u}_n(n, \gamma) = \frac{\left[ G \left( 1 - (-1)^n \right) + \delta_n^2 + \eta \delta_n^4 \right]}{\delta_n^2 q} \left( q^\beta + H_1 \right).$$

or equivalently we can write as:

$$\bar{u}_n(n, \gamma) = \frac{\left[ G \left( 1 - (-1)^n \right) + \delta_n^2 + \eta \delta_n^4 \right]}{\delta_n^2 q} \left( q^\beta + H_1 \right).$$

Applying the Laplace inverse to Eq. (14), we obtain the following result:

$$u_n(n, \tau) = \frac{\left[ G (1 - (-1)^n) + \delta_n^2 + \eta \delta_n^4 \right]}{\delta_n^2 q} \left( q^\beta + H_1 \right).$$

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Finally, our solution is the sum of the post-transient (steady state) solution and transient (unsteady) solution. Steady state and unsteady solutions are given as under:

$$u_p(\xi) = 1 - G - \frac{\left[ 1 - \frac{Gd}{2} \right]}{\frac{d}{2}} \xi - \frac{G \gamma^2}{2} + G \left( \frac{\cosh (\frac{\xi}{2} - \xi)}{\cosh (\frac{\xi}{2})} \right).$$
transient solutions are given by
\[ u_\tau (\xi, \tau) = 2 \sum_{n=0}^{\infty} \left[ \frac{\lambda_n}{}\right] \delta_n \left( \frac{\lambda_n}{}\right) \sin (\delta_n \xi). \]

(20)

IV. EXACT SOLUTIONS WITH CAPUTO-FABRIZIO DERIVATIVES

In order to use CF fractional model for generalized Couette flow of CSF, we replace partial derivatives of order one with respect to \( \tau \) by CF fractional operator of order \( \alpha \) Eq. (6), can be written in the following form:

\[ \text{CF} \text{D}_\alpha^\tau \text{Reu}(\xi, \tau) = G + \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} - \eta \frac{\partial^4 u(\xi, \tau)}{\partial \xi^4}, \]

(21)

where \( \text{CF} \text{D}_\alpha^\tau (.) \) is known as CF fractional derivatives with time fractional order of order \( \alpha \), which is defined as [37].

\[ \text{CF} \text{D}_\alpha^\tau (\tau) = \frac{1}{1 - \alpha} \int_0^\tau \exp \left( - \frac{\alpha (\tau - t)}{1 - \alpha} \right) f'(t) dt, \]

for \( 0 < \alpha < 1 \). (22)

Applying the Laplace transform to Eq. (21) and incorporate the given initial condition from Eq. (7), we get the following form:

\[ \frac{q \text{Reu}(\xi, q)}{q + H_1} = \frac{G}{q} + \frac{d^2 u(\xi, q)}{d \xi^2} - \eta \frac{d^4 u(\xi, q)}{d \xi^4}. \]

(23)

To apply sine Fourier transform we multiply both sides of the Eq. (23) by \( \sin \left( \frac{\pi \xi}{d} \right) \) and taking the limits of integration from 0 to \( d \) with respect to \( \xi \) and incorporate Eq. (12), we obtain:

\[ \bar{u}_n (n, q) = \left[ \frac{G(1 - (-1)^n)}{\delta_n^2 q(\lambda_n + \delta_n^2 + \eta \delta_n^4)} (q + M_1) \right] \]

\[ + \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} q \right] + \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} (q + M_2) \right] \]

(24)

or equivalently we can write as:

\[ \bar{u}_n (n, q) = \left[ \frac{G(1 - (-1)^n)}{\delta_n^2 q(\lambda_n + \delta_n^2 + \eta \delta_n^4)} (q + M_1) \right] \]

\[ + \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} q \right] - \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} (q + M_2) \right]. \]

(25)

Applying the Laplace inverse to Eq. (25), we obtain the following result:

\[ u_\tau (n, \tau) = \left[ \frac{G(1 - (-1)^n)}{\delta_n^2 q(\lambda_n + \delta_n^2 + \eta \delta_n^4)} \right] \]

\[ + \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} \right] \exp (-M_2 \tau), \]

(26)

where \( \lambda = \frac{1}{\alpha} \), \( \delta_n = \frac{\pi n}{d} \), \( M_1 = \frac{\alpha}{\delta_n} \) and \( M_2 = \frac{(\delta_n^2 + \eta \delta_n^4)M_1}{\lambda n + \delta_n^2 + \eta \delta_n^4} \). After some calculations, from Eq. (26), we can write as:

\[ u_\tau (n, \tau) = \frac{1 - (-1)^n}{\delta_n^2 q(\lambda_n + \delta_n^2 + \eta \delta_n^4)} \]

\[ + \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} \right] \exp (-M_2 \tau), \]

(27)

Now, applying the inverse sine-Fourier transform to Eq. (27) we obtain the following form [48], [49].

\[ u_\tau (\xi, \tau) = 1 - G - \left( \frac{1}{d} - \frac{Gd}{2} \right) \xi \exp \left( \frac{\frac{\pi \xi}{d} - \xi}{\frac{\pi \xi}{d} + \xi} \right) \]

\[ - G \left[ \frac{\cosh \left( \frac{\pi \xi}{d} - \xi \right)}{\cosh \left( \frac{\pi \xi}{d} + \xi \right)} \right] \]

\[ + \left[ \frac{G(1 - (-1)^n)}{\delta_n^2 q(\lambda_n + \delta_n^2 + \eta \delta_n^4)} \right] \exp (-M_2 \tau), \]

(28)

after some simplification we get the following solution:

\[ u_\tau (\xi, \tau) = 1 - G - \left( \frac{1}{d} - \frac{Gd}{2} \right) \xi \]

\[ - G \left[ \frac{\cosh \left( \frac{\pi \xi}{d} - \xi \right)}{\cosh \left( \frac{\pi \xi}{d} + \xi \right)} \right] \]

\[ + \frac{2}{d} \sum_{n=1}^\infty \left[ \frac{G(1 - (-1)^n)}{\delta_n (\lambda_n + \delta_n^2 + \eta \delta_n^4)} \right] \exp (-M_2 \tau). \]

(29)

Finally, our solution is the sum of the post-transient (steady state) solution and transient (unsteady) solution. Steady state and unsteady solutions are given as under:

\[ u_\tau (\xi) = 1 - G - \left( \frac{1}{d} - \frac{Gd}{2} \right) \xi \]

\[ - G \left[ \frac{\cosh \left( \frac{\pi \xi}{d} - \xi \right)}{\cosh \left( \frac{\pi \xi}{d} + \xi \right)} \right], \]

(30)
transient solutions are given by

$$u_t (\xi, \tau) = -\frac{2}{d} \sum_{n=1}^{\infty} \left[ \frac{G (1 - (-1)^n)}{+\delta_n^2 + \eta_n^4} \right] \frac{\text{Re}\lambda}{\delta_n^3 (1 + \delta_n^2 + \eta_n^4)} \times \sin (\delta_n \xi) \exp (-M_2 \tau). \quad (31)$$

V. LIMITING CASES

A. NEWTONIAN VISCOUS FLUID

In order to reduced couple stress fluid solutions to a Newtonian viscous fluid, we ignored the effect of couple stress parameter ($\eta = 0$) the governing equation (6) reduced to the following form:

$$\text{Re} \frac{\partial u(\xi, \tau)}{\partial \tau} = G + \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2}. \quad (32)$$

To develop the AB fractional model for Newtonian viscous fluid we replace partial derivatives with respect to $\tau$ and $\beta$ is the fractional operator of AB derivatives Eq. (32) can be written as:

$$\text{ABD}_\epsilon^\beta \text{Re} u(\xi, \tau) = G + \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2}. \quad (33)$$

Applying the Laplace transform to Eq. (33) and incorporate the given initial condition from Eq. (7), we get the following form:

$$q^\beta \text{Re} \bar{u}(\xi, q) = \frac{G}{q} + \frac{\partial^2 \bar{u}(\xi, q)}{d \xi^2}. \quad (34)$$

Apply sine Fourier transform to Eq. (34) we get the following result:

$$\bar{u}_n (n, q) = \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n^3 q} \left( q^\beta + \ell_1 \right) + \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n \left( \text{Re}\gamma + \delta_n^2 \right)} (-\ell_2)^\frac{1}{\beta} \left( q^\beta + \ell_2 \right), \quad (35)$$

or equivalently we can write as:

$$\bar{u}_n (n, q) = \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n^3 q} \left( q^\beta + \ell_1 \right) + \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n \left( \text{Re}\gamma + \delta_n^2 \right)} (-\ell_2)^\frac{1}{\beta} \left( q^\beta + \ell_2 \right). \quad (36)$$

Applying the Laplace inverse to Eq. (36), we obtain the following result:

$$u_n (n, \tau) = \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n^3} \left( q^\beta + \ell_1 \right) + \frac{[G (1 - (-1)^n) + \delta_n^2]}{\delta_n \left( \text{Re}\gamma + \delta_n^2 \right)} (-\ell_2)^\frac{1}{\beta} \left( q^\beta + \ell_2 \right) \times \sum_{k=0}^{\infty} \frac{(-\ell_2)^k}{\Gamma (k + 1 + \beta)} (-\ell_2)^\frac{1}{\beta} \left( q^\beta + \ell_2 \right). \quad (37)$$

where $\gamma = \frac{1}{\tau - \beta}, \delta_n = \frac{\pi n}{d}, \ell_1 = \frac{\beta}{\tau - \beta} \text{ and } \ell_2 = \frac{(\delta_n^2 \ell_1)}{\text{Re}\gamma + \delta_n^2}.$

Now, applying the inverse sine-Fourier transform to Eq. (37) we obtain the following form [48], [49].

$$u_\epsilon (\xi, \tau) = 1 - G - \frac{1}{d} \frac{G \xi^2}{2} \times \sin (\delta_n \xi) \exp (-M_2 \tau). \quad (38)$$

Steady state and unsteady solutions are given as under:

$$u_p (\xi) = 1 - G - \frac{1}{d} \frac{G \xi^2}{2} \times \sin (\delta_n \xi) \exp (-M_2 \tau). \quad (39)$$

B. CLASSICAL CSF SOLUTION

In order to reduced our solution, CSF fractional model to the classical CSF model we put $\beta \rightarrow 1$ then our solution reduces to the solution obtained by Akhtar, S., & Shah, N. A [44].

Using the following property:

$$\lim_{\beta \rightarrow 1} \text{ABD}_\epsilon^\beta u(\xi, \tau) = \lim_{\beta \rightarrow 1} L^{-1} \left[ L \text{ABD}_\epsilon^\beta u(\xi, \tau) \right]$$

$$= L^{-1} \left[ q^\beta \bar{u}(\xi, q) - u(\xi, 0) \right]$$

$$= L^{-1} \left[ q \bar{u}(\xi, q) - u(\xi, 0) \right]$$

$$= L^{-1} \left[ u'(\xi, \tau) \right] = u'(\xi, \tau). \quad (41)$$

$$u'(\xi, \tau) = 1 - G - \frac{1}{d} \frac{G \xi^2}{2} \times \sin (\delta_n \xi) e^{-(\delta_n^2 + \delta_2^2) \tau} \exp (-M_2 \tau). \quad (42)$$

The results obtained in Eq. (42) shows that our solution is reduced to the solutions obtained by [44]. Which shows the validity of our obtained solutions.
VI. SPECIAL CASE

In the absence of external pressure gradient \( (G = 0) \) the governing equation can be reduced in the following form:

\[
\text{AB} D^\beta \text{Reu}(\xi, \tau) = \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} - \eta \frac{\partial^4 u(\xi, \tau)}{\partial \xi^4},
\]

(43)

After Applying the Laplace and sine Fourier transforms we get the following transform solutions:

\[
\bar{u}_n(n, q) = \frac{[\delta_n^2 + \eta \delta_n^4]}{\delta_n (\delta_n^2 + \eta \delta_n^4) q} + \frac{[\delta_n^2 + \eta \delta_n^4]}{\delta_n (\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4) \left(-H_2 \eta \right)^{\frac{1}{\beta}} \left(q^\beta + H_2 \right)},
\]

(44)

where \( \gamma = \frac{1}{1-R} \), \( \delta_n = \frac{\eta n}{\beta} \), \( H_1 = \frac{\beta}{1-R} \) and \( H_2 = \frac{\delta_n^2 + \eta \delta_n^4}{\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4} \).

Now, applying the inverse Laplace and sine-Fourier transform to Eq. (44) we obtain the following form [48], [49].

\[
u(\xi, \tau) = 1 - \left(\frac{1}{d}\right) \xi + \frac{2}{d} \sum_{n=0}^{\infty} \left( \frac{\delta_n^2 + \eta \delta_n^4}{\delta_n (\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4) \left(-H_2 \eta \right)^{\frac{1}{\beta}} \left(q^\beta + H_2 \right) \times \left(\frac{H_2 \eta + \delta_n^2 + \eta \delta_n^4}{1((\delta_n^2 + \eta \delta_n^4) + \delta_n^2 + \eta \delta_n^4)} \right) \right) \sin(\delta_n \xi).
\]

(45)

VII. SKIN FRICTION

The expression for skin friction can be written as

\[
C_f(y, \tau) = \frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial y^3},
\]

(46)

skin friction for the lower plate is given by

\[
C_{f_l}(0, \tau) = \frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial y^3},
\]

(47)

and the corresponding skin friction for upper plate is given as

\[
C_{f_u}(d, \tau) = \frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial y^3}.
\]

(48)

VI. SPECIAL CASE

In the absence of external pressure gradient \( (G = 0) \) the governing equation can be reduced in the following form:

\[
\text{AB} D^\beta \text{Reu}(\xi, \tau) = \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} - \eta \frac{\partial^4 u(\xi, \tau)}{\partial \xi^4},
\]

(43)

After Applying the Laplace and sine Fourier transforms we get the following transform solutions:

\[
\bar{u}_n(n, q) = \frac{[\delta_n^2 + \eta \delta_n^4]}{\delta_n (\delta_n^2 + \eta \delta_n^4) q} + \frac{[\delta_n^2 + \eta \delta_n^4]}{\delta_n (\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4) \left(-H_2 \eta \right)^{\frac{1}{\beta}} \left(q^\beta + H_2 \right)},
\]

(44)

where \( \gamma = \frac{1}{1-R} \), \( \delta_n = \frac{\eta n}{\beta} \), \( H_1 = \frac{\beta}{1-R} \) and \( H_2 = \frac{\delta_n^2 + \eta \delta_n^4}{\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4} \).

Now, applying the inverse Laplace and sine-Fourier transform to Eq. (44) we obtain the following form [48], [49].

\[
u(\xi, \tau) = 1 - \left(\frac{1}{d}\right) \xi + \frac{2}{d} \sum_{n=0}^{\infty} \left( \frac{\delta_n^2 + \eta \delta_n^4}{\delta_n (\text{Re} \gamma + \delta_n^2 + \eta \delta_n^4) \left(-H_2 \eta \right)^{\frac{1}{\beta}} \left(q^\beta + H_2 \right) \times \left(\frac{H_2 \eta + \delta_n^2 + \eta \delta_n^4}{1((\delta_n^2 + \eta \delta_n^4) + \delta_n^2 + \eta \delta_n^4)} \right) \right) \sin(\delta_n \xi).
\]

(45)

VIII. RESULTS AND DISCUSSION

The exact solutions for the generalized couple stress fluid between two parallel plates are obtained via Laplace transform and Fourier sine transforms technique. The CSF flow is due to the constant pressure gradient between the two parallel plates, where the upper plate is fixed and the lower plate is moving with constant velocity \( U_0 \). The geometry of the problem is shown in figure 1. Generalized couple stress fluid in a channel has been considered using two recently proposed definitions of fractional derivatives namely, CF and AB fractional derivatives. Exact solutions of the present problem have been obtained for both the cases CF and AB fractional derivatives and compared their results graphically. Furthermore, the obtained solutions for AB fractional model of Couple stress fluid in a channel are shown graphically for various embedded parameters. The obtained solutions of couple stress fluid are reduced to the Newtonian viscous fluid and classical CSF velocity in limiting sense, and a comparison of the velocity of the CSF and Newtonian viscous fluid by using graphical analysis is given. Furthermore, the CSF velocity with AB fractional approach is compared with the classical CSF velocity using graphical analysis. In the absence of an external pressure gradient, CSF solutions are also investigated and displayed the obtained results through the graph. Furthermore, the present solution reduced to the solution obtained by Akhtar and Shah [44]. The comparison of these results is shown in table 1 and figure 2 which are quite identical and validate our work.

As the present article is focused to obtain two solutions for CF and AB fractional derivatives the comparison of which is shown in figure 3. It can be seen that for a smaller time \( \tau = 0.05 \) the velocity obtained for AB fractional approach is higher than the velocity obtained for CF fractional approach. If we take unit time \( \tau = 1 \) the magnitude of both the velocities for CF and AB fractional approach become equal. For the greater values of the time \( \tau = 2 \) and \( \tau = 3 \) the magnitude of couple stress fluid velocities identical at the center of the channel for both CF and AB fractional approach, while at the lower plate the magnitude of velocity for CF

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>Present result ( u(\xi, \tau) )</th>
<th>Result of Akhtar and Shah [44]</th>
</tr>
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<tr>
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<tr>
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<td>1.029</td>
<td>1.037</td>
</tr>
</tbody>
</table>
The approach is higher than AB fractional approach and near the upper plate, this behavior of velocity reverses. Comparison of CF and AB fractional approaches is found quite similar to the comparison obtained by Podlubny [35].

In all the figures the effect of AB fractional parameter $\beta$ is compared with the classical CSF velocity. Where $\beta = 1$ shows the classical velocity of CSF. From all the graphs it can be seen that the magnitude of classical velocity is lower than the fractional velocity. Furthermore, by increasing fractional parameter $\beta$ results in retardation in the CSF velocity. Figure 4 shows the influence of external pressure gradient $G$ on the couple stress fluid between two parallel plates (channel flow). From the figure, it is observed that couple stress fluid velocity increases by increasing the absolute value of the external pressure gradient. It is due to the fact that external pressure accelerates the velocity of the fluid.

Figure 5 displays the behavior of CSF velocity for different values of the Reynolds number $Re$. Reynolds number $Re$ shows the ratio of inertial forces to the viscous forces. By increasing $Re$ of CSF their produces turbulence in the fluid due to which the CSF becomes more viscous as a result CSF...
velocity decreases. Figure 6 displays the variation in CSF velocity in a channel for different values of the time. From the figure, it is observed that for short interval of time the magnitude of the velocity is lower, while for a long time the CSF velocity increases as the model of CSF is unsteady in the formulation assumptions.
Figure 7 depicts the variation in velocity profile for different values of couple stress parameter $\eta$. From this figure, it is noticed that increasing couple stress parameter $\eta$ fluid velocity retards, it is due to the fact that couple stresses oppose fluid flow in the channel. Furthermore, from this figure, it is observed that for $\eta = 0$ the case is Newtonian viscous fluid and the velocity of Newtonian viscous fluid is greater than CSF velocity.

Figure 8 shows the comparison of CSF velocity (in the presence and absence of external pressure gradient $G$). From this figure, it can be seen that velocity with $G$ has maximum value as compare to velocity in the absence of external pressure gradient $G$. This comparison shows that the absolute value of the external pressure gradient is responsible to accelerate the magnitude of CSF velocity.

Numerical results of Skin fraction for CSF velocity for the lower plate are shown in table 1. Skin fraction variation is presented in tabular form for different parameters of interest. Increasing external pressure gradient Skin fraction of SCF
velocity increases for CF approach and decreasing for classical and AB fractional approach. Increasing ReSkin fraction for CSF velocity increases for CF and AB approach and decreases for classical CSF velocity. Increasing time $t$ Skin fraction of CSF velocity decreases for CF, AB, and classical velocity. Increasing couple stress parameter $\eta$ Skin fraction of AB and CF increases while decreases Skin fraction of classical velocity. Increasing CF fractional operator $\alpha$ Skin fraction decreases for CF approach while increasing AB fractional operator $\beta$ results in an increase in the Skin fraction of the CSF velocity for AB fractional derivative.

In table 2 numerical results of Skin fraction for CSF velocity for the upper plate are shown. Variation in Skin fraction can be observed for AB, CF and classical CSF velocity and presented in tabular form for different parameters. Increasing external pressure gradient $G$ results in an increase in the Skin fraction of SCF velocity for AB and CF fractional derivatives and classical CSF. Increasing $Re$, skin fraction for CSF velocity decreases for CF and AB approach and classical CSF velocity. Increasing time Skin fraction of CSF velocity increases for CF, AB, and classical velocity. Increasing couple stress parameter $\eta$ Skin fraction of AB and CF and classical CSF velocity increases. Increasing CF fractional operator $\alpha$ and AB fractional operator $\beta$ result in a decrease in the Skin fraction of the CSF velocity for CF and AB fractional approach.
The new idea of Atangana–Baleanu fractional derivative has been applied for the first time to Couple Stress Fluid flow between two parallel plates under the effect of the external pressure gradient. CF derivatives solutions are also provided for the sake of comparison and verification of the new scheme of AB for CSF. For a unit time, both AB and CF shows similar results while for smaller and greater times variation in CSF velocity is observed.

The following results are obtained during the solutions to the problem:

- The effect of AB and CF fractional derivatives are shown in the velocity profile. From the graphical results, we noticed that the magnitude of both the velocities are identical at the unit time and variation is observed for a large and small time.
- Increasing external pressure gradient velocity of couple stress fluid increases.
- The velocity of the couple stress fluid decreases by increasing the values of Re.
- Increasing the time t velocity of the fluid increases.
- Results for Newtonian fluid is obtained in limiting sense from CSF.

REFERENCES


