Adaptive control and synchronization of a coupled dynamo system with uncertain parameters

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Abstract

This paper treats the adaptive control and synchronization of the coupled dynamo system with unknown parameters. Based on the Lyapunov stability technique, an adaptive control laws are derived such that the coupled dynamo system is asymptotically stable and the two identical dynamo systems are asymptotically synchronized. Also the update rules of the unknown parameters are derived. Finally, numerical simulation of the controlled and synchronized systems are presented.

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1. Introduction

Chaos synchronization has received a great attention since 1990, due to its potential applications in different areas such as chemical reaction, biological systems, communications and others [1,2]. A chaotic system has complex dynamical behaviors that process some special features, such as being extremely sensitive to any variations of initial conditions, having bounded trajectories in the phase space and other [3].

Chaos, an interesting phenomenon in nonlinear dynamical systems, has been developed and thoroughly studied over the past two decade. A chaotic system is a nonlinear deterministic system that displays complex, noisy-like and unpredictable behavior. The sensitive dependence on the initial condition and the system parameters variations is a prominent characteristic of chaotic behavior. Research efforts have studied the control chaos and chaos synchronization problems in many physical chaotic systems see for example [2,4–7].

Recently, after the pioneering work of Ott et al. [6], several control strategies for stabilizing chaos have been proposed [13,14]. There are two main approaches for controlling chaos: non-feedback control and feedback control. The concept of chaos synchronization involves making two chaotic systems which oscillate in a synchronized manner. At present, synchronization is studied primarily either between two similar systems which vary slightly in their parameters, or between two completely different systems.

Generally speaking, the synchronization phenomenon has the following feature: the trajectories of the drive and response systems are identical notwithstanding starting from different initial conditions. However, slight errors of initial

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conditions, for chaotic dynamical systems, will lead to completely different trajectories. Therefore, how to control of two chaotic systems to be synchronized has seen a flurry of research activities for over a decade. Many approaches have been presented for the synchronization of chaotic systems such as linear and nonlinear feedback control [9–11]. Most of them are based on the exactly knowing of the system structure and parameters. But in practice, some or all of the system’s parameters are unknown. Moreover, these parameters change from time to other. A lot of works have preceded to solve this problem using adaptive synchronization [12,13]. Several different approaches, including some conventional linear control techniques and advanced nonlinear control schemes, have already been successfully applied to the above problems. However, other used the optimal control techniques see for example [8,15–17].

In the present paper, the problem of adaptive control and adaptive synchronization for the chaos synchronization of two identical dynamo system with uncertain two parameters is introduced. It is proved that the adaptive synchronization can be achieved using feedback control law. Using the Lyapunov technique, the control laws that achieve global asymptotic stability of both the dynamo and error dynamical system and updating rules of estimates of the unknown system parameters are derived. It is found that the feedback control ensures the adaptive synchronization of the adopted system is nonlinear. Further, numerical simulation study is included to show the obtained results.

2. System of coupled dynamos

The system of coupled dynamos capable of chaotic behavior consists of a set of dynamos connected together so that the current generated by any one of them produces the magnetic field for another. Taking for simplicity, the case where there are only two dynamos, we denote the angular velocities of their rotors by \( \omega_1, \omega_2 \) and the currents generated by \( x_1, x_2 \), respectively. Then, with appropriate normalization of variables, the dynamical equations can be described by the following set of ordinary differential equations:

\[
\begin{align*}
\dot{x}_1 &= -\mu_1 x_1 + \omega_1 x_2, \\
\dot{x}_2 &= -\mu_2 x_2 + \omega_2 x_1, \\
\dot{\omega}_1 &= -q_1 - \epsilon_1 \omega_1 - x_1 x_2, \\
\dot{\omega}_2 &= q_2 - \epsilon_2 \omega_2 - x_1 x_2,
\end{align*}
\]

(2.1)

where \( q_1 \) and \( q_2 \) are the torques applied to the rotors, and \( \mu_1, \mu_2, \epsilon_1 \) and \( \epsilon_2 \) are positive constants representing dissipative effects. The model is thus of fourth order, but we can simplify it considerably by setting \( q_1 = q_2 = 1, \epsilon_1 = \epsilon_2 = 0 \) and \( \mu_1 = \mu_2 = \mu \). It then follows that \( \dot{\omega}_1 = \dot{\omega}_2 \). Therefore, we can write

\[
\begin{align*}
\dot{x}_1 &= x_3 + z, \\
\dot{x}_2 &= x_3 - z,
\end{align*}
\]

(2.2)

where \( z \) is a constant of the motion. Finally, the dynamical equations of the coupled dynamos can be written in the following simple form:

\[
\begin{align*}
\dot{x} &= -\mu x + (z + x)y, \\
\dot{y} &= -\mu y + (z - x)x, \\
\dot{z} &= 1 - xy.
\end{align*}
\]

(2.3)

The divergence equation of the system (2.3) is given by

\[
\nabla \cdot \mathbf{\Phi} = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z} = -2\mu < 0,
\]

(2.4)

where the vector function \( \mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3) \) is given by \( \mathbf{\Phi} = (-\mu x + zy, -\mu y + zx - xy, 1 - xy) \).

Then, system (2.3) is a forced dissipative system similar to Lorenz system. But it is different from Lorenz system. Thus, the solutions of the system (2.3) are bounded as \( t \to \infty \) for positive values of both \( \mu \) and \( x \). But in a sense defined by Vanek and Celikovsky [18] the Lorenz system satisfies the condition, \( a_{12}a_{21} > 0 \), while the coupled dynamos dynamical system satisfies the condition, \( a_{12}a_{21} < 0 \) where \( a_{ij} \) are the elements of the constant system matrix of their linear parts \( A = [a_{ij}] \). Hence, the coupled dynamos system and the Lorenz system are different types of system.

The equilibrium points of the system (2.3) are \( E_1 = (\beta_1, \beta_2, \gamma) \), \( E_2 = (-\beta_1, -\beta_2, \gamma) \), where \( \beta_i (i = 1, 2) \) and \( \gamma \) are given by

\[
\beta_1 = \sqrt{\frac{\gamma + \gamma}{\mu}}, \quad \beta_2 = \sqrt{\frac{\gamma - \gamma}{\mu}}, \quad \gamma = \sqrt{x^2 + \mu^2}.
\]

(2.5)

Since the model is unaltered by reversing the signs of, both singular points have the same dynamical properties. The Jacobian of the linearized system about the first equilibrium point \( E_1 \) is
The eigenvalues of this matrix are

\[ \lambda_1 = -2\mu, \quad \lambda_{2,3} = \frac{-2\gamma}{\mu} \]

It follows that the linearized system has a transient term that decays to zero as \( t \) increases (since \( \lambda_1 < 0 \)) plus an oscillatory solution. It is therefore naturally stable in the limit. The same result is true at the other equilibrium points. Consequently, we cannot conclude anything about the behavior of the dynamical system (2.3) by means of linearization. However, a numerical integration of these equations is quite revealing. It shows that the equilibrium are actually unstable and that the orbits encircle one of them a number of times before switching suddenly to encircle the other equilibrium where it oscillates for awhile before again switching rapidly back about the original point. The orbits are never captured by either equilibrium point, and the limiting behavior is apparently chaotic since the number of oscillations in the neighborhood of each equilibrium point is unpredictable [19].

The following figures display the dynamo chaotic attractors, the phase coordinates and projections of the trajectory for different values the system parameters \( \alpha, \mu \) with different initial states.

**Fig. 1** displays the dynamo chaotic attractors for the system parameters \( \alpha = 2.5, \mu = 1.5, \alpha = 5.9, \mu = 3; \alpha = 10, \mu = 2 \), when the dynamo system is subjected the initial states \((x(0), y(0), z(0)) = (1.2, 2.1, 3)\), \((x(0), y(0), z(0)) = (-1.2, 2.1, 4)\), and \((x(0), y(0), z(0)) = (-12.2, 5.1, 3)\) respectively.

**Figs. 2 and 3** display both of dynamo phase trajectory components and trajectory projections on \( xy, yz, xz \) planes respectively for the system parameters \( \mu = 0.5 \) and \( \alpha = 5.9 \) when this system is subjected to the initial state \((x(0), y(0), z(0)) = (10.3, 5.5, 2.1)\).

In the next sections, we will apply the Lyapunov technique to study the problems of adaptive control and synchronization of two identical dynamo systems with uncertain two parameters feedback control approach.

### 3. Adaptive control of dynamo system

For the purpose of adaptive control of dynamo system with uncertain parameters by a feedback control approach, let us first assume that we have the controlled coupled system in the following form:

\[
\begin{align*}
\dot{x}_1 &= -\mu x_1 + (x_3 + \hat{x}) x_2 + u_1, \\
\dot{x}_2 &= -\mu x_2 + (x_3 - \hat{x}) x_1 + u_2, \\
\dot{x}_3 &= 1 - x_1 x_2 + u_3,
\end{align*}
\]

Fig. 1. (a–c) Dynamo chaotic attractor.
where $u_1$, $u_2$ and $u_3$ are external control inputs that will be suitably designed to derive the trajectory of the system specified by the equilibrium states $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ to any of these equilibrium points of the uncontrolled system.

In what follows we will study the adaptive stabilization of the equilibrium states $x_i = \bar{x}_i$, $u_i = 0$, $i = 1, 2, 3$, $\dot{x} = x$, $\dot{\mu} = \mu$ and the update rule of the estimates $\hat{\mu}(t)$, $\hat{x}(t)$ of the system unknown parameters.

To obtain the equations of perturbed states of dynamo system (3.1) about the equilibrium state (3.2), we introduce the following new variables:

$$\xi(t) = x_i(t) - \bar{x}_i \quad (i = 1, 2, 3).$$

Substituting (3.3) into (3.1) taking into account the identities that satisfy by the equilibrium states, we get the following system:

$$\begin{align*}
\dot{\xi}_1 &= -\mu \xi_1 + (\bar{x}_3 + \dot{x}) \xi_2 + (\bar{x}_2 + \xi_2) \xi_3 + u_1, \\
\dot{\xi}_2 &= -\mu \xi_2 + (\bar{x}_3 - \dot{x}) \xi_1 + (\bar{x}_1 + \xi_1) \xi_3 + u_2, \\
\dot{\xi}_3 &= -(\bar{x}_1 + \xi_1) \xi_2 - \bar{x}_2 \xi_1 + u_3.
\end{align*}$$

This system can be satisfied by the special solution

$$\xi_i = 0, \quad u_i = 0, \quad \dot{\mu} = \mu, \quad \dot{x} = x,$$
which we will asymptotically stabilized using the Lyapunov technique with the help of the feedback control inputs $u_i$ ($i = 1, 2, 3$).

**Theorem 3.1.** If the following nonlinear feedback control law

$$
\begin{align*}
    u_1 &= - (k_1 - \mu) \xi_1 - \xi_2 \xi_3, \\
    u_2 &= - (k_2 - \mu) \xi_2 - \xi_1 \xi_3, \\
    u_3 &= - k_3 \xi_3 - \xi_1 \xi_2 \\
\end{align*}
$$

(3.6)

and the update rule of the estimates $\hat{\mu}$ and $\hat{x}$ of the unknown system parameters $\mu$ and $x$ is given by

$$
\begin{align*}
    \dot{\hat{\mu}} &= - l_1 (\hat{\mu} - \mu) + \xi_1^2 + \xi_2^2, \\
    \dot{\hat{x}} &= - l_2 (\hat{x} - x),
\end{align*}
$$

(3.7)

where $k_i$ are positive real constant and $l_i$ ($i = 1, 2$) are nonnegative real constants, are adopted. Then the solution (3.5) is globally asymptotically stable.

**Proof.** The proof of this theorem is based on the choice of the following quadratic Lyapunov function

$$
2V = \sum_{i=1}^{3} \xi_i^2 + (\hat{x} - x)^2 + (\hat{\mu} - \mu)^2.
$$

(3.8)

The time derivative of the Lyapunov function $V$ along the trajectory of the systems (3.4) and (3.7) taking into consideration the feedback control law (3.6), takes the form

$$
\dot{V} = - [k_1 \xi_1^2 + k_2 \xi_2^2 + k_3 \xi_3^2 + l_1 (\hat{\mu} - \mu)^2 + l_2 (\hat{x} - x)^2] \leq 0.
$$

(3.9)

The time derivative $\dot{V}$ is negative definite if and only if $l_i > 0$ ($i = 1, 2$), in this case the solution (3.5) is asymptotically stable in the Lyapunov sense. On other hand when $l_1 = l_2 = 0$, the time derivative $\dot{V}$ becomes only negative semi-definite function (constant sign function). Therefore, in this case under the action of the feedback controllers (3.6) the solution (3.5) is only stable but not necessary asymptotic. At first, it appears that the proposed control law falls, but in fact this control law is successful after all, as the following more refined analysis shows.

Consider the case $l_1 = l_2 = 0$. Therefore, the time derivative of the Lyapunov function takes the form

$$
\dot{V} = - (k_1 \xi_1^2 + k_2 \xi_2^2 + k_3 \xi_3^2) \leq 0.
$$

(3.10)

It is clear that the derivative $\dot{V}$ is zero if and only if $\xi_1 = \xi_2 = \xi_3 = 0$ holds whether $\hat{\mu} = \mu$, $\hat{x} = x$ or not. If for some particular time instant, say $T$, $V(T) = 0$, and if further $V(t) = 0$ for all $t \geq T$, then of course we have $V(t) = \text{const.}$, but not necessarily zero for all $t \geq T$.

Assume that $V(T) = 0$. What must be established this contingency cannot occur, namely we must prove that $V$ cannot remain zero for all $t$ and it becomes zero only at the zero solution (3.5). It is clear that the derivative $\dot{V}$ is zero if and only if $\xi_1 = \xi_2 = \xi_3 = 0$ holds. But in this case we have $u_1 = u_2 = u_3 = 0$ and directly we find that $\dot{\hat{\mu}} = \mu$, $\dot{\hat{x}} = x$, therefore the function $\dot{V} = 0$ only at the zero solution (3.5). Then, in the case $l_1 = l_2 = 0$ the solution (3.5) is also asymptotically stable.

Hence, the nonlinear feedback controllers (3.6) globally asymptotically stabilizes the solution (3.5) of the systems (3.4) and (3.7). \( \square \)

4. **Adaptive synchronization of the dynamo system**

To study the adaptive synchronization of two identical dynamo systems with uncertain parameters by a feedback control approach, let us first assume that we have two dynamo systems the first is the derive system and it has the state variables $x_i(t)$ ($i = 1, 2, 3$) with positive real parameters $\mu$ and $x$, and the second is the response and it has the state variables $y_i(t)$ ($i = 1, 2, 3$) with parameters $\hat{\mu}(t)$ and $\hat{x}(t)$ that represent the estimates of the unknown parameters $\mu$, $x$, respectively of the response system, that is

$$
\begin{align*}
    \dot{x}_1 &= - \mu x_1 + (x_3 + x)x_2, \\
    \dot{x}_2 &= - \mu x_2 + (x_3 - x)x_1, \\
    \dot{x}_3 &= 1 - x_1 x_2,
\end{align*}
$$

(4.1)
and

\[
\begin{align*}
\dot{y}_1 &= -\mu y_1 + (y_3 + \hat{\alpha})y_2 + u_1, \\
\dot{y}_2 &= -\mu y_2 + (y_3 - \hat{\alpha})y_1 + u_2, \\
\dot{y}_3 &= 1 - y_1y_2 + u_3,
\end{align*}
\]

where \(u_i (i = 1, 2, 3)\) are control inputs which will be suitably designed to achieve the trajectories of both derive and response will be synchronized.

Subtracting Eqs. (4.1) from (4.2) yields the error dynamical system between the response and derive systems

\[
\begin{align*}
\dot{e}_1 &= -(\hat{\mu} - \mu)x_1 + (\hat{\alpha} - \alpha)x_2 - \hat{\mu}e_1 + (x_3 + \hat{\alpha})e_2 + e_3e_3 + x_2e_3 + u_1, \\
\dot{e}_2 &= -(\hat{\mu} - \mu)x_2 + (\hat{\alpha} - \alpha)x_1 - \hat{\mu}e_2 + (x_3 - \hat{\alpha})e_1 + e_1e_3 + x_1e_3 + u_2, \\
\dot{e}_3 &= -\hat{e}_1e_2 - x_2e_1 - x_1e_2 + u_3,
\end{align*}
\]

where \(e_i = y_i - x_i (i = 1, 2, 3)\).

To study the problem of adaptive synchronization of two identical dynamo systems with uncertain parameters we formulate the following theorem.

**Theorem 4.1.** If the following nonlinear feedback control law:

\[
\begin{align*}
u_1 &= -(k_1 - \hat{\mu})e_1 - x_3e_2, \\
u_2 &= -(k_2 - \hat{\mu})e_2 - x_3e_1, \\
u_3 &= -k_3e_3 - e_1e_2
\end{align*}
\]

and the update rule of the estimators \(\hat{\mu}\) and \(\hat{\alpha}\) of the unknown system parameters \(\mu\) and \(\alpha\) is given by

\[
\begin{align*}
\dot{\hat{\mu}} &= -l_1(\hat{\mu} - \mu) + x_1e_1 + x_2e_2, \\
\dot{\hat{\alpha}} &= -l_2(\hat{\alpha} - \alpha) - x_2e_1 + x_1e_2,
\end{align*}
\]

where \(k_i (i = 1, 2)\) are positive real constant and \(l_i (i = 1, 2)\) are nonnegative real constants, are adopted. Then, the two identical dynamo system (4.1) and (4.2) will be asymptotically synchronized.

**Proof.** Since both of the error system (4.3) and the update system parameters (4.5) admit the special solution

\[
e_i = 0, \quad u_i = 0, \quad \hat{\mu} = \mu, \quad \hat{\alpha} = \alpha.
\]

Clearly, the adaptive synchronization problem is now replaced by equivalent problem of stabilizing the system (4.3) and (4.5) about the equilibrium states (4.6) using the control law (4.6). Therefore, the proof of the asymptotic stability of this solution is similar of the asymptotic stability of the solution (3.5) as given in Theorem 3.1. Consequently, the asymptotic synchronization of two dynamo systems is proved. □

In the next section we will study the numerical solution of the resulting controlled system and adaptive synchronized system with different values of both system parameters and initial states.

**5. Numerical simulation**

The numerical simulation seems necessary since the resulting controlled and synchronized systems are nonlinear systems of differential equations. This section displays graphically the numerical integration of the controlled, derive, response, error and updating system parameters for different values of the system parameters and control gains parameters.

For better understanding the behavior of the dynamo adaptive control and synchronization mechanism, we have applied the method of Lyapunov technique and we will examined the numerical solution with different cases by varying both of the system parameters and the control gains parameters with the initial state

\[
(\xi_1(0), \xi_2(0), \xi_3(0)) = (2, 2.5, 4), \quad (\hat{\mu}(0), \hat{\alpha}(0)) = (0.2, 2),
\]

when the dynamo controlled system is subjected to the feedback control (3.6).
Fig. 4 displays the perturbation \((\zeta_1(t), \zeta_2(t), \zeta_3(t))\) of the dynamo state variables, and the estimates \(\hat{\mu}(t), \hat{\alpha}(t)\) of the dynamo system for the system parameters are \(\mu = 3, \alpha = 2\) and the control parameters are \(k_1 = 3, k_2 = 2, k_3 = 4, l_2 = 5l_1 = 5\).

Fig. 5 displays the perturbation \((\zeta_1(t), \zeta_2(t), \zeta_3(t))\) of the dynamo state variables, and the estimates \(\hat{\mu}(t), \hat{\alpha}(t)\) of the dynamo system for the system parameters are \(\mu = 2, \alpha = 5\) and the control parameters are \(k_1 = 50, k_2 = 10, k_3 = 40, l_1 = 0.5, l_2 = 0.5\).

In what follows we display a comparison between both the derive and response dynamo system components; synchronization error and the estimates of the response system parameters for the initial states

\[
(x_1(0), x_2(0), x_3(0)) = (2, 2.5, 4), \quad (y_1(0), y_2(0), y_3(0)) = (1, 2, 1.5), \quad (\hat{\mu}(0), \hat{\alpha}(0)) = (0.2, 0.2). 
\] (5.2)

Fig. 6 displays the dynamo derive and corresponding response trajectory components \((x_i(t), y_i(t))\), the synchronization error \(e_i(t)\) and estimates \(\hat{\mu}, \hat{\alpha}\) of the response system parameters against time respectively for \(\mu = 2.2, \alpha = 1.5, k_1 = 5.3, k_2 = 8.1, k_3 = 5.4, l_1 = 3.5, l_2 = 2.5\).

Fig. 7 displays the dynamo derive and corresponding response trajectory components \((x_i(t), y_i(t))\), the synchronization error \(e_i(t)\) and estimates \(\hat{\mu}, \hat{\alpha}\) of the response system parameters against time respectively for \(\mu = 20.2, \alpha = 10.5, k_1 = 5.3, k_2 = 8.1, k_3 = 5.4, l_1 = 3.5, l_2 = 2.5\).

Fig. 8 displays the dynamo derive and corresponding response trajectory components \((x_i(t), y_i(t))\), the synchronization error \(e_i(t)\) and estimates \(\hat{\mu}, \hat{\alpha}\) of the response system parameters against time respectively for \(\mu = 20.2, \alpha = 10.5, k_1 = 15.3, k_2 = 18.1, k_3 = 5.4, l_1 = 30.5, l_2 = 20.5\).

Finally, we observe that the response trajectories fully converge to the derive and the estimates of the response system parameters also converge to the exact values of the derive system parameters. Further, the synchronization errors tend to zero for arbitrary initial state of the error system.
Fig. 6. (a) Response and derive first component, (b) response and derive second component, (c) response and derive third component, (d) synchronization errors and (e) estimates of the system parameters.

Fig. 7. (a) Response and derive first component, (b) response and derive second component, (c) response and derive third component, (d) synchronization errors and (e) estimates of the system parameters.
6. Conclusion

The problems of adaptive control and synchronization of two dynamo systems with uncertain parameters are studied using Lyapunov technique. The update rule of the unknown parameters is obtained in both cases. Also the feedback control laws are obtained using the conditions of asymptotic stability. Important numerical simulation is presented.

References


Fig. 8. (a) Response and derive first component, (b) response and derive second component, (c) response and derive third component, (d) synchronization errors and (e) estimates of the system parameters.