Mathematical Modeling of Optimum 3 Step Stress Accelerated Life Testing for Generalized Pareto Distribution

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To cite this article:

Abstract: This article contains the optimum 3 step stress accelerated life test under cumulative exposure model. The lifetimes of test units are assumed to follow a generalized Pareto distribution. The scale parameter of the used failure time distribution at the constant stress level is assumed to have a log-linear and quadratic relationship with the stress. A comparison between linear plan and quadratic plan by maximum likelihood estimators for the different sample sizes is shown in the table. The optimum test plans is obtained by minimizing the asymptotic variance of the maximum likelihood estimator of the $100^{th}$ percentile of the lifetime distribution at normal stress condition for the model parameters. Tables of optimum times of changing stress level for both plans are also obtained.

Keywords: Accelerated Life Testing, Generalized Pareto Distribution, Asymptotic Variance, Maximum Likelihood Estimate, Life Stress Relationship (Linear and Quadratic), Cumulative Exposure Model

1. Introduction

Due to the fast development in the technology, the manufacturing plan is continuously improving. For this reason it is difficult to obtain information about lifetime of products or materials with high reliability at the time of testing under normal conditions because testing under normal operating conditions require a very long period of time and need an extensive number of units under test. So it is generally very expensive and impractical to complete reliability testing under normal conditions. Hence, to handle these problems, the study of accelerated life test (ALT) has been developed. ALT makes it possible to quickly obtain information on the life distribution of products by inducing early failure with higher than normal stress. There are mainly three types of life test methods in accelerated life testing design. The first method is constant stress ALT, second is step-stress ALT and the third is Progressive stress ALT. In step stress accelerated life testing, the test items are subjected to successively higher levels of stress at pre-assigned test times. The level of stress is increased step by step until all items have failed or the test stops for other reasons.

generalized K-H model by assuming that the lifetime of a test unit follows a Weibull distribution for type-I censored data. Alhadeed and Yang (2005) considered a simple SSALT plan for optimal time of changing stress level using the Log-Normal distribution.

2. The Model and Test Procedure

2.1. Generalized Pareto (GP) Distribution

The generalized Pareto (GP) distribution is also known as the Pareto of second kind with two parameters or Lomax distribution. This distribution has been widely used in the field of reliability modeling and life testing. Bandera-Zahrahi (2012) presented the maximum likelihood estimation for generalized Pareto distribution under progressive censoring with binomial removals. The life times of the test items is said to have the generalized Pareto distribution if it has the probability density function (PDF)

\[ f(y) = \alpha \theta (1 + \theta y)^{-(\alpha + 1)} \quad y > 0, \alpha, \theta > 0 \quad (1) \]

where \( \theta \) is the scale parameter and \( \alpha \) is the shape parameter of the distribution.

Now, the cumulative distribution function is given by

\[ F(y) = 1 - (1 + \theta y)^{-\alpha} \quad y > 0, \alpha, \theta > 0 \quad (2) \]

The survival function of the GP distribution is given by

\[ F(y) = (1 + \theta y)^{-\alpha} \quad y > 0, \alpha, \theta > 0 \quad (3) \]

And the corresponding hazard rate is given by

\[ h(y) = \frac{\alpha \theta}{(1 + \theta y)} \]

2.2. SSALT with Cumulative Exposure Model

According to the cumulative exposure model, the CDF in SSALT for 3-step is given by

\[ F(t) = \begin{cases} \frac{F_S(t)}{S_1} & 0 \leq t < \tau_1 \\ \frac{F_S(t - \tau_1 + S_1)}{S_1} & \tau_1 \leq t < \tau_2 \\ \frac{F_S(t - \tau_1 + S_2)}{S_1} & \tau_2 \leq t < \infty \end{cases} \quad (4) \]

with \( S_i \) the solution of \( F_S(S_i) = F_S(\tau_i) \) and \( S_2 \) the solution of \( F_S(\tau_2 - \tau_1 + S_1) = F_S(\tau_2 - \tau_1 + S_1) \). On solving we get

\[ S_1 = \frac{\theta_1}{\theta_2 - \theta_1} \tau_1 \quad \text{and} \quad S_2 = \frac{\theta_2}{\theta_3 - \theta_2} \left( \tau_2 - \tau_1 \right) + \frac{\theta_1}{\theta_3 - \theta_1} \tau_1 \]

Hence Generalized cumulative exposure model for 3 step stress is as follow

\[ F(t) = \begin{cases} 1 - (1 + \theta t)^{\alpha} & 0 \leq t < \tau_1 \\ 1 - \left[ 1 + \theta_1 (t - \tau_1 + \theta_1 \tau_1) \right]^{\alpha} & \tau_1 \leq t < \tau_2 \\ 1 - \left[ 1 + \theta_1 (t - \tau_2 + \theta_1 \tau_2) + \theta_2 (\tau_2 - \tau_1) + \theta_1 \tau_2 \right]^{\alpha} & \tau_2 \leq t < \infty \end{cases} \]

The corresponding PDF for the cumulative exposure model is given as above by equation \((5)\)

\[ f(t) = \begin{cases} \alpha \theta (1 + \theta t)^{-(\alpha+1)} & 0 \leq t < \tau_1 \\ \alpha \theta_1 [1 + \theta_1 (t - \tau_1 + \theta_1 \tau_1)]^{-(\alpha+1)} & \tau_1 \leq t < \tau_2 \\ \alpha \theta_1 [1 + \theta_1 (t - \tau_2 + \theta_1 \tau_2) + \theta_2 (\tau_2 - \tau_1) + \theta_1 \tau_2]^{-(\alpha+1)} & \tau_2 \leq t < \infty \end{cases} \quad (5) \]

2.3. Assumptions

1. Testing is performed at the three stress levels \( S_1, S_2 \) and \( S_3 \), where \( S_1 > S_2 > S_3 \).
2. The lifetime of the test units follow a generalized Pareto distribution under any stress.
3. The parameter \( \alpha \) is independent of time and stress.
4. The scale parameter \( \theta \) at stress level \( i, i = 1, 2, 3 \) is a log linear function of stress given by \((i)\) or \((ii)\)

\[ \log \theta_i = a + b S_i \quad (i) \]

\[ \log \theta_i = a + b S_i + c S_i^2 \quad (ii) \]

where, \( a, b \) and \( c \) are unknown parameters on the nature of the product and the test method.

A random sample of \( n \) identical units are initially placed on low stress \( S_1 \) and run until pre-specified time \( \tau_1 \) when the stress is changed to high stress \( S_2 \) for those remaining units that have not failed. The test is continued until pre-specified time \( \tau_2 \) when stress is changed to \( S_3 \), and continued until all remaining units fail.

3. Likelihood Function

In order to obtain the MLE of the model parameters, let \( t_{ij} \), \( i = 1, 2, 3, j = 1, 2, 3, \ldots, n_i \) be the observed failure time of a test unit \( j \) under the stress level \( i \), where \( n_i \) denotes the number of units failed at the stress level \( i \). The likelihood function is given by

\[ L(t, \alpha, \theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} \left[ \alpha \theta_1 (1 + t_{i_1 j} \theta_1)^{-(\alpha+1)} \right] \prod_{j=1}^{n_2} \left[ \alpha \theta_2 \left( 1 + (t_{i_2 j} - \tau_1) \theta_2 + \theta_1 \theta_2 \right)^{-(\alpha+1)} \right] \prod_{j=1}^{n_3} \left[ \alpha \theta_1 \left( 1 + (t_{i_3 j} - \tau_2) \theta_1 + (\tau_2 - \tau_1) \theta_2 + \theta_1 \theta_2 \right)^{-(\alpha+1)} \right] \quad (6) \]
Log likelihood function of equation (6) can be written as

\[ l = n \log \alpha + n_1 \log \beta_1 + n_2 \log \beta_2 + n_3 \log \beta_3 \]

\[ - (\alpha + 1) \sum_{j=1}^{n_1} \log \left( 1 + t_{j1} \beta_1 \right) + \sum_{j=1}^{n_2} \log \left[ 1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1 \right] \]

\[ + \sum_{j=1}^{n_3} \log \left[ 1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1 \right] \]

\[ l = n \log \alpha + n_1 \left( a + b \tau_1 \right) + n_2 \left( a + b \tau_2 \right) + n_3 \left( a + b \tau_3 \right) \]

\[ - (\alpha + 1) \sum_{j=1}^{n_1} \log \left( 1 + t_{j1} e^{a+b \tau_1} \right) + \sum_{j=1}^{n_2} \log \left[ 1 + \left( t_{j2} - \tau_1 \right) e^{a+b \tau_2} + \tau_1 e^{a+b \tau_1} \right] \]

\[ + \sum_{j=1}^{n_3} \log \left[ 1 + \left( t_{j3} - \tau_2 \right) e^{a+b \tau_3} + \left( \tau_2 - \tau_1 \right) e^{a+b \tau_2} + \tau_1 e^{a+b \tau_1} \right] \]

4. Optimum Linear Step-Stress Test

Using the relation \( \log \beta_i = a + b \tau_i, i = 1, 2, 3 \), the log likelihood function becomes

\[ l = n \log \alpha + n_1 \left( a + b \tau_1 \right) + n_2 \left( a + b \tau_2 \right) + n_3 \left( a + b \tau_3 \right) \]

\[ - (\alpha + 1) \sum_{j=1}^{n_1} \log \left( 1 + t_{j1} e^{a+b \tau_1} \right) + \sum_{j=1}^{n_2} \log \left[ 1 + \left( t_{j2} - \tau_1 \right) e^{a+b \tau_2} + \tau_1 e^{a+b \tau_1} \right] \]

\[ + \sum_{j=1}^{n_3} \log \left[ 1 + \left( t_{j3} - \tau_2 \right) e^{a+b \tau_3} + \left( \tau_2 - \tau_1 \right) e^{a+b \tau_2} + \tau_1 e^{a+b \tau_1} \right] \] (7)

MLEs of \( a \) and \( b \) are obtained by solving the equations \( \frac{\partial l}{\partial a} = 0 \) and \( \frac{\partial l}{\partial b} = 0 \).

\[ \frac{\partial l}{\partial a} = n - (\alpha + 1) \sum_{j=1}^{n_1} \frac{t_{j1} \beta_1}{1 + t_{j1} \beta_1} + \sum_{j=1}^{n_2} \frac{\left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1}{1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1} + \sum_{j=1}^{n_3} \frac{\left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \] (8)

\[ \frac{\partial l}{\partial b} = n S_1 + n_2 S_2 + n_3 S_3 \]

\[ - (\alpha + 1) \sum_{j=1}^{n_1} \frac{S_1 t_{j1} \beta_1}{1 + t_{j1} \beta_1} + \sum_{j=1}^{n_2} \frac{S_2 \left( t_{j2} - \tau_1 \right) \beta_2 + S_1 t_{j1} \beta_1}{1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1} + \sum_{j=1}^{n_3} \frac{S_3 \left( t_{j3} - \tau_2 \right) \beta_3 + S_2 \left( \tau_2 - \tau_1 \right) \beta_2 + S_1 t_{j1} \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \] (9)

\[ \frac{\partial^2 l}{\partial a^2} = - (\alpha + 1) \sum_{j=1}^{n_1} \frac{t_{j1} \beta_1}{1 + t_{j1} \beta_1} + \sum_{j=1}^{n_2} \frac{\left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1}{1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1} + \sum_{j=1}^{n_3} \frac{\left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \] (10)

\[ \frac{\partial^2 l}{\partial b^2} = - (\alpha + 1) \sum_{j=1}^{n_1} \frac{S_1^2 t_{j1} \beta_1}{1 + t_{j1} \beta_1} + \sum_{j=1}^{n_2} \frac{S_2^2 \left( t_{j2} - \tau_1 \right) \beta_2 + S_1^2 t_{j1} \beta_1}{1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1} + \sum_{j=1}^{n_3} \frac{S_3^2 \left( t_{j3} - \tau_2 \right) \beta_3 + S_2^2 \left( \tau_2 - \tau_1 \right) \beta_2 + S_1^2 t_{j1} \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \]

\[ \left[ \frac{S_2^2 \left( t_{j3} - \tau_2 \right) \beta_3 + S_1^2 \left( \tau_2 - \tau_1 \right) \beta_2 + S_2^2 \left( t_{j3} - \tau_2 \right) \beta_3 + S_1^2 t_{j1} \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \right] \] (11)

\[ \frac{\partial^2 l}{\partial a \partial b} = - (\alpha + 1) \sum_{j=1}^{n_1} \frac{S_1 t_{j1} \beta_1}{1 + t_{j1} \beta_1} + \sum_{j=1}^{n_2} \frac{S_2 \left( t_{j2} - \tau_1 \right) \beta_2 + S_1 t_{j1} \beta_1}{1 + \left( t_{j2} - \tau_1 \right) \beta_2 + \tau_1 \beta_1} + \sum_{j=1}^{n_3} \frac{S_3 \left( t_{j3} - \tau_2 \right) \beta_3 + S_2 \left( \tau_2 - \tau_1 \right) \beta_2 + S_1 t_{j1} \beta_1}{1 + \left( t_{j3} - \tau_2 \right) \beta_3 + \left( \tau_2 - \tau_1 \right) \beta_2 + \tau_1 \beta_1} \] (12)

The Fisher information matrix is obtained by taking the negative second partial derivatives of the log-likelihood function. Fisher Information matrix is given by
The log of the $100p^{th}$ percentile of the lifetime $t_p(S_0)$ at the design stress $S_0$ is given by

$$\hat{\xi}(S_0) = \log(t_p(S_0)) = a + bS_0 + \log\left(\frac{1-P}{\theta_i}\right)$$

The Asymptotic variance is given by

$$AV_1(\hat{\xi}(S_0)) = AV_1(\log(t_p(S_0))) = AV_1\left(\hat{a} + \hat{b}S_0 + \log\left(\frac{1-P}{\theta_i}\right)\right) = H_{F_1}^{-1}H'$$

where $H = \begin{bmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a \partial b} & \frac{\partial^2 l}{\partial a \partial c} \\ \frac{\partial^2 l}{\partial b \partial a} & \frac{\partial^2 l}{\partial b^2} & \frac{\partial^2 l}{\partial b \partial c} \\ \frac{\partial^2 l}{\partial c \partial a} & \frac{\partial^2 l}{\partial c \partial b} & \frac{\partial^2 l}{\partial c^2} \end{bmatrix}$ and $F_1$ is the asymptotic fisher information matrix.

Therefore, the optimal times $\tau_1$ and $\tau_2$ to change stress levels under different values of stresses and sample sizes will be obtained numerically using equation (13) which minimizes $AV_1(\hat{\xi}(S_0))$.

### 5. Optimum Quadratic Step-Stress Test

For the quadratic model, using the log linear relationship

$$\log \theta_i = a + bS_1 + cS_i^2, \; i = 1, 2, 3.$$  \hspace{1cm} (14)

For this quadratic model, the testing can be done using a 3-step, step-stress ALT. Thus, the log likelihood function (7) can be extended to the quadratic model after substituting the value of $\theta_i$ from equation (14) given by

$$l = n\log \alpha + n_1(a + bS_1 + cS_1^2) + n_2(a + bS_2 + cS_2^2) + n_3(a + bS_3 + cS_3^2) - (\alpha + 1)\left[\sum_{j=1}^n \log \left[1 + t_{1j}e^{a + bS_j + cS_j^2}\right]\right] + \sum_{j=1}^n \log \left[1 + \left(t_{2j} - \tau_1\right)e^{a + bS_j + cS_j^2} + \tau_1e^{a + bS_j + cS_j^2}\right] + \sum_{j=1}^n \log \left[1 + \left(t_{3j} - \tau_2\right)e^{a + bS_j + cS_j^2} + \tau_2e^{a + bS_j + cS_j^2}\right]$$  \hspace{1cm} (15)

MLEs of $a, b$ and $c$ are obtained by solving the equations $\frac{\partial l}{\partial a} = 0, \frac{\partial l}{\partial b} = 0$ and $\frac{\partial l}{\partial c} = 0$.

$$\frac{\partial l}{\partial a} = n\left(-\alpha + 1\right)\sum_{j=1}^n \frac{t_{1j}^\theta_i}{1 + t_{1j}^\theta_i} + \sum_{j=1}^n \frac{t_{1j} - \tau_1}{1 + t_{1j} - \tau_1} + \sum_{j=1}^n \frac{t_{2j} - \tau_1}{1 + t_{2j} - \tau_1} + \sum_{j=1}^n \frac{t_{3j} - \tau_1}{1 + t_{3j} - \tau_1}$$

$$\frac{\partial l}{\partial b} = n_1S_1 + n_2S_2 + n_3S_3 - \left(-\alpha + 1\right)\sum_{j=1}^n \frac{S_1t_{1j}^\theta_i}{1 + t_{1j}^\theta_i} + \sum_{j=1}^n \frac{S_1(t_{1j} - \tau_1)}{1 + t_{1j} - \tau_1} + \sum_{j=1}^n \frac{S_1(t_{2j} - \tau_1)}{1 + t_{2j} - \tau_1} + \sum_{j=1}^n \frac{S_1(t_{3j} - \tau_1)}{1 + t_{3j} - \tau_1}$$

$$\frac{\partial l}{\partial c} = n_1S_1^2 + n_2S_2^2 + n_3S_3^2 - \left(-\alpha + 1\right)\sum_{j=1}^n \frac{S_1^2t_{1j}^\theta_i}{1 + t_{1j}^\theta_i} + \sum_{j=1}^n \frac{S_1^2(t_{1j} - \tau_1)}{1 + t_{1j} - \tau_1} + \sum_{j=1}^n \frac{S_1^2(t_{2j} - \tau_1)}{1 + t_{2j} - \tau_1} + \sum_{j=1}^n \frac{S_1^2(t_{3j} - \tau_1)}{1 + t_{3j} - \tau_1}$$
\[
\frac{\partial^2 l}{\partial a^2} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{(t_j - \tau_1) \theta_j + \tau_1 \theta_j}{1 + (t_j - \tau_1) \theta_j + \tau_1 \theta_j} + \sum_{j=1}^{n} \frac{(t_j - \tau_2) \theta_j + (\tau_2 - \tau_1) \theta_2 + \tau_1 \theta_2}{1 + (t_j - \tau_2) \theta_2 + (\tau_2 - \tau_1) \theta_2 + \tau_1 \theta_2} \right] \]

\[
\frac{\partial^2 l}{\partial b^2} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{S_j^2 t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{S_j^2 (t_j - \tau_1) \theta_j + S_j^2 \tau_1 \theta_j + (S_j^2 + S_j^2 - 2S_j S_j) \theta_2 \theta_j \tau_1 (t_j - \tau_1)}{1 + (t_j - \tau_1) \theta_2 + \tau_1 \theta_2} \right]
\]

\[
\frac{\partial^2 l}{\partial c^2} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{S_j^2 t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{S_j^2 (t_j - \tau_2) \theta_j + S_j^2 \tau_2 \theta_j + (S_j^2 + S_j^2 - 2S_j S_j) \theta_2 \theta_j \tau_1 (t_j - \tau_2)}{1 + (t_j - \tau_2) \theta_2 + \tau_2 \theta_2} \right]
\]

\[
\frac{\partial^2 l}{\partial ab} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{S_j^2 t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{S_j^2 (t_j - \tau_1) \theta_2 + S_j^2 \tau_1 \theta_2 + S_j^2 + S_j^2 - 2S_j S_j \theta_2 \theta_1 (t_j - \tau_1)}{1 + (t_j - \tau_1) \theta_2 + \tau_1 \theta_2} \right]
\]

\[
\frac{\partial^2 l}{\partial ac} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{S_j^2 t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{S_j^2 (t_j - \tau_2) \theta_2 + S_j^2 \tau_2 \theta_2 + S_j^2 + S_j^2 - 2S_j S_j \theta_2 \theta_1 (t_j - \tau_2)}{1 + (t_j - \tau_2) \theta_2 + \tau_2 \theta_2} \right]
\]

\[
\frac{\partial^2 l}{\partial bc} = - (\alpha + 1) \left[ \sum_{j=1}^{n} \frac{S_j^2 t_j \theta_j}{1 + t_j \theta_j} + \sum_{j=1}^{n} \frac{S_j^2 (t_j - \tau_1) \theta_2 + S_j^2 \tau_1 \theta_2 + S_j^2 + S_j^2 - 2S_j S_j \theta_2 \theta_1 (t_j - \tau_1)}{1 + (t_j - \tau_1) \theta_2 + \tau_1 \theta_2} \right]
\]

Fisher Information matrix is given by

\[
F_2 = \begin{bmatrix}
-\frac{\partial^2 l}{\partial a^2} & -\frac{\partial^2 l}{\partial ab} & -\frac{\partial^2 l}{\partial ac} \\
-\frac{\partial^2 l}{\partial ab} & -\frac{\partial^2 l}{\partial b^2} & -\frac{\partial^2 l}{\partial bc} \\
-\frac{\partial^2 l}{\partial ac} & -\frac{\partial^2 l}{\partial bc} & -\frac{\partial^2 l}{\partial c^2}
\end{bmatrix}
\]

change time \( \tau_1 \) and \( \tau_2 \). The log of the 100P\textsuperscript{th} percentile of the lifetime \( t_p(S_0) \) at the design stress \( S_0 \) is given as

\[
\hat{S}(S_0) = \log \left( t_p(S_0) \right) = a + bS_0 + cS_0^2 + \log \left( \frac{(1-P)}{\theta_i} \right)
\]

The asymptotic variance at the design stress \( S_0 \) is then given by

Here the Optimum criterion is to obtain optimum stress
\[
AV_2 \left( \hat{\xi} (S_0) \right) = HF_2^{-1} H'
\]

(25)

Where \( H = \begin{bmatrix}
\frac{\partial \hat{\xi} (S_0)}{\partial a} & \frac{\partial \hat{\xi} (S_0)}{\partial b} & \frac{\partial \hat{\xi} (S_0)}{\partial c} & \frac{\partial \hat{\xi} (S_0)}{\partial d}
\end{bmatrix} \) and \( F_2 \) is the fisher information matrix. Therefore, the optimal times \( \tau_1 \) and \( \tau_2 \) will be obtained numerically using equation (25) which minimizes \( AV_2 \left( \hat{\xi} (S_0) \right) \).

Table 1. The Maximum likelihood estimate and Mean Absolute Error for \( a=0.0087 \).

<table>
<thead>
<tr>
<th>n</th>
<th>Parameter</th>
<th>Quadratic case</th>
<th>Linear case</th>
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<td></td>
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6. Simulation Study

A numerical study was conducted in order to investigate the existence of the optimal stress change points and to evaluate them as a function of varying parameters. Simulations are performed to investigate the performances of the MLEs through their mean absolute error (MAE) for both relationships. Comparison between both plans is shown by calculating efficiencies. Table 1 present the Maximum likelihood estimates for \( n=20, 60, 80, 100 \) and 120 and their respective Mean absolute Error for both Quadratic and Linear relationship. Table 2 presents the results of optimal design of step-stress ALT for different sized samples and finally Table 3 Compound Linear Test-Plan Efficiencies.

Table 2. The results of optimal design of step-stress ALT for different sized sample.

<table>
<thead>
<tr>
<th>n</th>
<th>a=0.69 b=1.25</th>
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<td>nA V</td>
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<td>( \tau_2 )</td>
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<td>60</td>
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Table 3. Compound Linear Test Plan Efficiencies

<table>
<thead>
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<th>S3</th>
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<th>Efficiencies</th>
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<td>3.58</td>
<td>4.58</td>
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<tr>
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7. Conclusion

The present article deals with parameter estimation of generalized Paretodistribution under 3 step stress ALT plan. The objective is to plan a test that achieves the best reliability estimates. Two types of relationship are assumed between scale parameter and Stress. One links scale parameter linearly with stress while other have quadratic relationship. Comparison between both is shown by calculating estimates and their respective error. Efficiencies for both plans are calculated for different level of stress. Apart from that the results of optimal design of step-stress ALT for different sample size is shown. The performance of step-stress testing plans and model assumptions are generally evaluated by the properties of the maximum likelihood estimates of model parameters. Estimates of quadratic are more stable with relatively small Mean absolute error as sample size increases. Maximum likelihood estimators are consistent and asymptotically normally distributed. In short, it may be concluded that the present step stress ALT plan works well.
and has a good choice to be considered in the field of accelerated life testing.

References


